

THE STRUCTURE OF SELF-DUAL PLANE GRAPHS WITH MAXIMUM DEGREE 4

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ABSTRACT. Self-dual plane graphs have been studied extensively. C. A. B Smith and W. T. Tutte published *A class of self-dual maps* in 1950 [9]; in 1992, Archdeacon and Richter [1] described a method for constructing all self-dual plane graphs and a second construction was produced by Servatius and Christopher [5] in 1992. Both constructions are inductive. In this paper, we produce four templates from which all self-dual plane graphs with maximum degree 4 (self-dual spherical grids) can be constructed. The self-dual spherical grids are further subdivided into 27 basic automorphism classes. Self-dual spherical grids in the same automorphism class have similar architecture. A smallest example of each class is constructed.

1. INTRODUCTION

The literature on constructing and classifying self-dual plane graphs is rather extensive. In addition to the papers mentioned in the abstract, there several other relevant papers. The paper by Servatius and Servatius [8] uses matroid methods to construct all self-dual graphs. Other related constructions can be found in [2] by G. Brinkmann, S. Greenberg, C. Greenhill, B McKay, R. Thomas and P. Wollan and in [4] by M. Dutour and M. Deza. The papers of Servatius and Servatius, [6] and [7], uses the symmetry structure to classify self-dual graphs. The feature that distinguishes our approach from these inductive constructions is the set of four templates: each self-dual plane graph with maximum degree 4 is constructed directly by choosing the values for the parameters in the appropriate template. Furthermore one may specify the symmetry structure in the selection of template and parameter values. By a *spherical grid, or an SG*, we shall mean a plane graph $\Gamma = (V, E, F)$ with vertex and face degrees restricted to 3 and 4.

Lemma 1. *Let Γ be a spherical grid with v_3 vertices of degree-3 and f_3 faces of degree-3. Then v_3 and f_3 are even numbers and $v_3 + f_3 = 8$.*

Proof. Let v_4 denote the number of vertices of degree 4 in Γ , let f_4 denote the number of faces of degree 4 and let e denote the number of edges. Then:

- (i) $3v_3 + 4v_4 = 2e$;
- (ii) $3f_3 + 4f_4 = 2e$;
- (iii) $(v_3 + v_4) - e + (f_3 + f_4) = 2$.

Equalities (i) and (ii) are obtained by summing the vertex and face degrees; the third equation is Euler's formula for plane graphs. To maintain parity in the first two equations, v_3 and f_3 must be even numbers. Subtracting the first and second equations from 4 times the third gives $v_3 + f_3 = 8$. \square

The proof of the next lemma is also straight forward and left to the reader.

Lemma 2. *The dual of a spherical grid is another spherical grid.*

We denote the infinite plane graph of the rectangular tessellation of the plane by Λ . One of the easiest ways to describe a spherical grid is to cut it along some straight line segments and unfold it as a polygon in Λ . We illustrate this in the left-hand diagram of Figure 1 below:

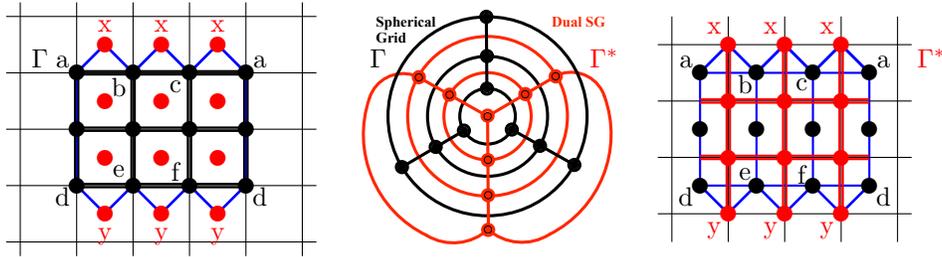


FIGURE 1. On the left, the black spherical grid is unfolded and embedded in Λ . It has $v_3 = 6$, $v_4 = 3$, $f_3 = 2$ and $f_4 = 6$. Shifting this diagram in Λ by $\frac{1}{2}$ unit up and to the right gives an unfolding of its dual (red) spherical grid in Λ (pictured on the right).

We may then reconstruct the spherical grid from this polygon by cutting out the polygon and identifying corresponding outer edges. Folding the left-hand polygon along the $b-e$ and $c-f$ segments, identifying the $a-d$ segments and then folding down the small triangles at either end reconstructs this spherical grid. One very nice feature of this diagram is that by shifting it one half unit to the right and one half unit up (pictured on the right-hand grid), we have its unfolded dual. A similar folding of the right-hand polygon yields the dual spherical grid.

A spherical grid is said to be a *triangle free spherical grid* or a *TFSG* if it has eight degree-3 vertices and no triangular faces. With the following construction, we may associate a TFSG with any spherical grid Γ . Draw the dual spherical grid Γ^* (in red) superimposed on the original spherical grid Γ (in black). Bisect each edge in the spherical grid by its dual adding, as a vertex, the intersection of the crossing edges. In Figure 2, we illustrate this construction with the SG from Figure 1. We easily see that each triangular face is replaced by three quadrilateral faces surrounding a (dual) vertex of degree-3 and each quadrilateral face is replaced by four quadrilateral faces surrounding a (dual) vertex of degree 4. We also note that the new vertices, the intersections of edges and dual edges, all have degree 4. We call these the *mixed vertices* and color them white. Hence the resulting graph is a TFSG with all eight of its degree-3 vertices among the vertices colored red and black. The nice feature is that this process is reversible.

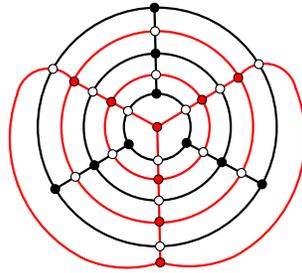


FIGURE 2. The construction of the triangle free spherical grid (TFSG) for the spherical grid (SG) in Figure 1.

Let $\Gamma = (V, E, F)$ be any TFSG. Observe that, since each face is a quadrilateral, the graph is bipartite. If the eight degree-3 vertices do not belong to the same cell of the bipartition, Γ did not come from a spherical grid by the above construction. If all eight degree-3 vertices do belong to the same cell of the bipartition, it could be constructed from a SG and we are able to reverse the construction. Assume that $\Gamma = (V, E, F)$ is a TFSG with all degree-3 vertices in the same class of its bipartition (see the first diagram in Figure 3). Color the vertices in the cell devoid of degree-3 vertices white and, for now, color the remaining vertices green (see the second diagram in Figure 3). We must now partition these green vertices into two cells, red vertices and black vertices. To accomplish this we construct an auxiliary graph. Each face of Γ is a square with the green and white vertices alternating around its boundary. We add a diagonal edge in each face joining the two green vertices in its boundary (see the second diagram in Figure 3).

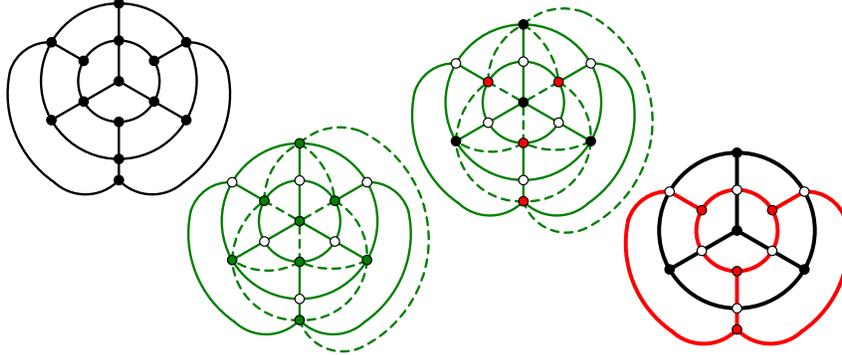


FIGURE 3. The construction of a dual pair of spherical grids from a triangle free spherical grid.

Our auxiliary graph Ω will have as its vertex set the green vertices and as its edge set these new diagonal edges. Since Γ plus diagonals is planar, Ω is planar. Furthermore, each face of Ω is a square with a white vertex in its center. It follows that Ω is bipartite. We arbitrarily color the vertices in one cell red and the vertices in the other cell black (see the third diagram in Figure 3). Since the black and red vertices alternate around a face of Ω , the neighbors of each white vertex in Γ alternate red and black around that vertex. Hence in Γ , each of its edges has one endpoint colored white while the other endpoint is colored either red or black. An edge of Γ is now colored red if it has a red endpoint and black if it has a black endpoint (see the last diagram in Figure 3). A TFSG with such a vertex and edge coloring is called a *properly colored* TFSG.

Finally, let Γ be a properly colored TFSG and consider the subgraph of black and white vertices and black edges. All white vertices have degree 2 and are the central vertex of a path of length 2 joining two black vertices. Replacing these paths by single edges results in (the black) spherical grid which we denote by Γ_b . Each face has a single red vertex in its center and the face is square if the red vertex has degree 4 and triangular if the red vertex has degree 3. Hence, the similarly constructed red SG, Γ_r , is the dual of Γ_b and Γ is the TFSG corresponding to either the black or red SG.

All of our spherical grids are plane graphs. We call an automorphism of a plane graph a *symmetry*. Almost all of the SGs we consider will be 3-connected and hence all graph automorphisms will be symmetries. However, a few small SSGs will not be 3-connected and the distinction between automorphism and symmetry will be necessary. Let Γ be a properly colored TFSG. Since Γ is a connected bipartite graph, any symmetry of Γ must preserve or interchange cells of the bipartition.

The white vertices form one partition class and the union of the red and black vertices the other. Since there are no white vertices of degree-3, all symmetries must map the class of white vertices onto itself. Any symmetry of Γ induces a symmetry on the auxiliary graph Ω and, since Ω is connected and bipartite, that symmetry must either preserve color (map black vertices to black vertices and red to red) or reverse the colors. Hence, a symmetry of Γ either induces a symmetries on both Γ_b and Γ_r or a graph isomorphism between Γ_b and Γ_r .

Lemma 3. *Let Γ be the TFSG constructed from the SG Δ . Then Δ is self-dual if and only if Γ admits a color-reversing symmetry.*

Proof. We may assume that $\Delta = \Gamma_b$, the black SG of Γ . Then Γ_r , the red SG of Γ , equals Δ^* . Any color-reversing symmetry γ of Γ maps the vertices of Γ_b onto the vertices of Γ_r and corresponds to an isomorphism between Δ and Δ^* . Conversely assume that there is an isomorphism from Δ to Δ^* . That isomorphism extends to an isomorphism of Γ , as the TFSG constructed on Γ_b , onto Γ , as the TFSG constructed on Γ_r , and hence to a color-reversing symmetry of Γ . \square

From now on we will restrict our attention to *special spherical grids* (SSGs) defined as follows: a properly colored TFSG, $\Gamma = (V, E, F)$, that admits a color-reversing symmetry. An example of an SSG is pictured on the right in Figure 3. In fact this is the simplest example possible: here both Γ_b and Γ_r are copies of the tetrahedral graph. Visualizing Γ as drawn on the sphere, the antipodal map is one of the several color-reversing symmetries of Γ .

2. COORDINATE SYSTEMS ON AN SSGS

In order to investigate the special spherical grids, we alter Λ , the rectangular tessellation of the plane, as follows: we identify the vertices of Λ with the set of points in \mathcal{R}^2 that have integer coordinates. Integer points with both coordinates even are colored black; points with both coordinates odd are colored red and the remaining points with integer coordinates are colored white. Edges on horizontal lines with even y-intercept and edges on vertical lines with even x-intercept are colored black; all other edges are colored red. In constructing our templates in Λ , we will use the coordinate system just described. However, once the template is extracted and the identifications made these Λ -coordinates no longer make sense. Hence we introduce a system of *SSG-coordinates* relating vertices.

We use the term *segment* for the straight line segment joining two vertices. The Λ -coordinates of a directed segment are simply the coordinates of the vector from the first vertex to the second vertex. The SSG-coordinates of the segment are obtained as follows: choose a new coordinate system for Λ with one endpoint of the segment at the origin and the other endpoint on the positive x -axis or in the first quadrant. The Λ -coordinates of the second endpoint are then the SSG-coordinates of the segment: (x, y) or simply (x) if the second endpoint is on the x -axis. See Figure 4.

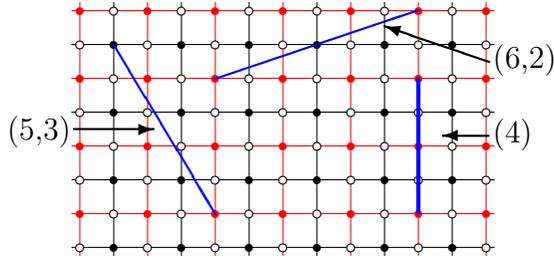


FIGURE 4. The assignment of coordinates to segments in Λ .

The advantages of these SSG-coordinates are that they are independent of the orientation of the segment and are well-defined on the SSG. Furthermore if u and w and u' and w' are pairs of points in Λ , then there is a symmetry of Λ mapping the first pair onto the second pair if and only if the segments joining pairs have the same or reversed SSG-coordinates. For later reference, we state this in the next easily verified lemma.

Lemma 4. *Let Γ be a SSG, let u and w be any two points so that there are no degree-3 vertices (other than possibly u or w) on any shortest path from u to w and let (p, q) be the SSG-coordinates of the segment joining u and w . Let ϕ be a symmetry of Γ . The SSG coordinates of the segment joining $\phi(u)$ and $\phi(w)$ are (p, q) when ϕ is orientation-preserving and (q, p) when ϕ is orientation-reversing.*

When we need to convert between Λ -coordinates and SSG-coordinates, we will refer to the following easily proved lemma.

Lemma 5. *Let u and v be two vertices of Λ and let (x, y) be the Λ -coordinates of the vector from v to u . Then the SSG-coordinates (p, q) of the segment joining v and u are computed as follows:*

- (i) $(p, q) = (x, y)$ when $x > 0$ and $y > 0$;
- (ii) $(p, q) = (y, -x)$ when $x < 0$ and $y > 0$;
- (iii) $(p, q) = (-y, x)$ when $x > 0$ and $y < 0$;

(iv) $(p, q) = (-x, -y)$ when $x < 0$ and $y < 0$.

Furthermore, if $xy = 0$, the SSG-coordinates (p) is the absolute value of the non-zero Λ -coordinate.

There are two natural ways to measure lengths for segments: the *graph theory length* of a segment with SSG-coordinates (p, q) is simply $p + q$ while the *geometric length* of a segment with SSG-coordinates (p, q) is $\sqrt{p^2 + q^2}$. Given a vertex we will frequently consider the set of degree-3 vertices *nearest* to it. By that we will mean those degree-3 vertices that are joined to the given vertex by the graph-theoretical shortest segments among the geometrically shortest segments. For example, the vertices joined to the vertex v by segments with SSG-coordinates $(10, 15)$, $(1, 18)$ and $(6, 17)$ all are a geometric distance of $\sqrt{325}$ from v , but the vertex at the end of the segment with SSG-coordinates $(1, 18)$ is nearest to v with a graph theory distance of 19; on the other hand, vertices joined to v by segments with SSG-coordinates $(1, 12)$, $(5, 8)$ and $(6, 7)$ all are a graph theory distance of 13 from v , but the vertex at the end of the segment with SSG-coordinates $(6, 7)$ is nearest to v with a geometric distance of $\sqrt{85}$. The following observation is clear and is labeled a lemma for easy reference.

Lemma 6. *Let Γ be an SSG and let v be a vertex. Then the all segments to the nearest degree-3 neighbors of v have SSG-coordinates (p, q) or (q, p) for a fixed p and q .*

3. REGIONS OF AN SSG

By definition all of the faces of an SSG are squares. Hence when the SSG is “cut open” and flattened out into a polygon on Λ (its *template*), the number of faces of the SSG is simply the total area of its template. Based on this observation we will be able to compute the number of vertices in a self-dual spherical grid in terms of the area of the template of the corresponding SSG.

Lemma 7. *Let $\Gamma = (V_b \cup V_r \cup V_w, E, F)$ be an SSG and let A denote the number of faces of Γ or the area of its template. Then $|V_b| = |V_r| = \frac{1}{4}A + 1$.*

Proof. Counting edges from the red-black side of the bipartition $[V_b \cup V_r, V_w]$ gives $|E| = (4|V_b| - 4) + (4|V_r| - 4) = 4(|V_b| + |V_r| - 2)$. On the other hand each face is bounded by 4 edges; so, $2|E| = 4|F|$. Combining these two equations yields $2(|V_b| + |V_r| - 2) = |F|$. Finally since Γ_b is self-dual $|V_b| = |V_r|$ and we have $4|V_b| - 4 = |F| = A$. \square

One basic region that will occur regularly in our investigation is that of a parallelogram. Specifically, most of our SSG templates will be partitioned into parallelograms with degree-3 vertices at the vertices of the parallelograms. Each parallelogram \mathcal{P} can be embedded in Λ with one vertex v_0 at the origin. Let $v_1 = (x_1, y_1)$ and $v_3 = (x_3, y_3)$ be the vertices of \mathcal{P} adjacent to v_0 . Then from basic vector geometry, we have that v_2 , the vertex opposite v_0 , has Λ -coordinates $(x_1 + x_3, y_1 + y_3)$. The matrix $\begin{bmatrix} x_1 & x_3 \\ y_1 & y_3 \end{bmatrix}$ gives the transformation of the plane that maps the unit square onto the parallelogram. Hence the absolute value of its determinate, $|x_1y_3 - x_3y_1|$, is the area of the parallelogram. If $x_1y_3 - x_3y_1$ is positive then the transformation preserves orientation and v_1, v_3 are in counterclockwise order about v_0 ; if $x_1y_3 - x_3y_1$ is negative then v_1, v_3 are in clockwise order.

A second method of partitioning an SSG will be into a “cylinder” sandwiched between two “caps.” Let Γ be an SSG and let v_0, v_1, v_2 and v_3 be four degree-3 vertices that when plotted relative to one another in Λ have no other degree-3 vertices of Γ in their convex hull \mathcal{H} . We may assume that v_0, v_1, v_2 and v_3 occur in counter clockwise order around the boundary of \mathcal{H} or v_1, v_2 and v_3 occur in counterclockwise order with v_0 interior to \mathcal{H} . In this case we may assume that the angle $\angle v_1v_0v_3$ is greater than 90° since v_0 has degree-3 in Γ . We may choose a ray from v_0 that separates v_1 and v_3 . Insert a 90° wedge with vertex v_0 and sides matching the ray, converting v_0 temporarily into a degree-4 vertex. In doing this, we see that in either case \mathcal{H} may be embedded as a convex region in Λ with vertices v_0, v_1, v_2 and v_3 in counterclockwise order around its boundary. Now we consider this region embedded as a convex quadrilateral in Λ .

Using an elementary geometric argument one easily sees that, given any convex quadrilateral, we may choose 90° wedges at the vertices so that the bounding rays across each edge of the quadrilateral are parallel. Then deleting each wedge and identifying its boundary rays creates a degree-3 vertex at each of v_0, v_1, v_2 and v_3 . The resulting alteration of Λ is an infinite cylinder with a “cap” containing \mathcal{H} at the end. A typical configuration is pictured in Figure 5. Segments of constant slope joining a point on the cylinder to itself are all parallel and have the same length. One of these segments connects one of the v_0, v_1, v_2 and v_3 to itself and is called the *rim of the cylinder*; the region it bounds is called the *cap*. This rim is also called *rim of the cap*. Consider the cap \mathcal{C} and its rim \mathcal{R} of the SSG and consider the vertex among the four remaining degree-3 vertices that is closest to \mathcal{R} . The segment parallel to \mathcal{R} through this vertex is then the rim of

the “bottom” cap and the region between the two rims is the actual *cylinder* of this decomposition of the SSG. Finally, if the cylinder is slit open along a segment connecting a degree-3 vertex on one rim to a degree-3 vertex on the other rim and unrolled we see that it is a parallelogram.

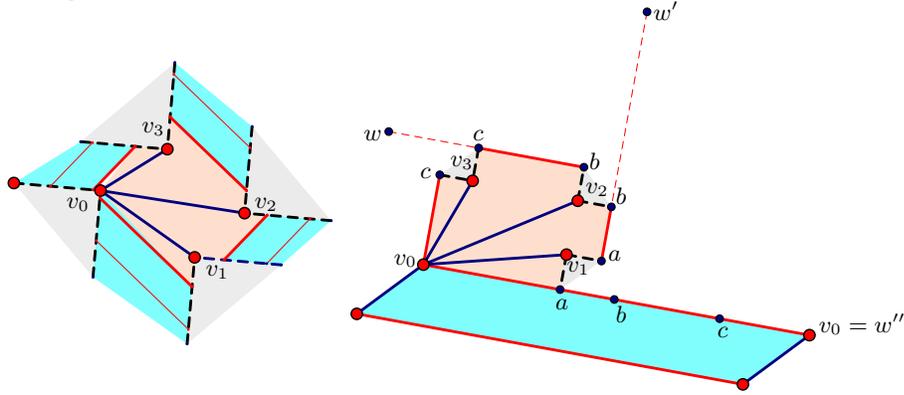


FIGURE 5. A Cap and Its Cylinder.

To compute the SSG-coordinates of the rim, embed the cap in Λ and, assuming that v_0 is the degree-3 vertex on the rim, place it at the origin. Assume that the Λ -coordinates of v_i are (x_i, y_i) and that v_1 is in the first quadrant. Now let w denote the image of the origin under the 90° clockwise rotation about v_3 , let w' denote the image of w under the 90° clockwise rotation about v_2 and let w'' denote the image of w' under the 90° clockwise rotation about v_1 . One easily checks that the Λ -coordinates of w'' are

$$(x_1 - y_1 + x_2 + y_2 - x_3 + y_3, x_1 + y_1 - x_2 + y_2 - x_3 - y_3).$$

When the wedges are removed and the cylinder reconstituted the segment $v_0 - w''$ becomes the rim of the cap. The above coordinates in the SSG-coordinate format are called the *rim coordinates* of the cylinder or cap. Computing the area of the cap is rather complicated. However, there are only two special cases for which we will actually use this cap-cylinder-cap decomposition, and we will compute the cap area only for these two cases. The two cases are: caps about the vertices of parallelograms and caps with rims on 45° lines; representatives of these two types pictured in Figure 6.

First consider the parallelogram (on the left in the figure). Substituting $x_1 + x_3$ for x_2 and $y_1 + y_3$ for y_2 in the rim formula gives $(2x_1 + 2y_3, 2y_1 - 2x_3)$ for the rim coordinates in this case. To compute the cap area, we start with the parallelogram itself; it has area $x_1y_3 - y_1x_3$. Next observe that the four triangular regions of the cap outside of the parallelogram (labeled A, B, C and D) may be assembled

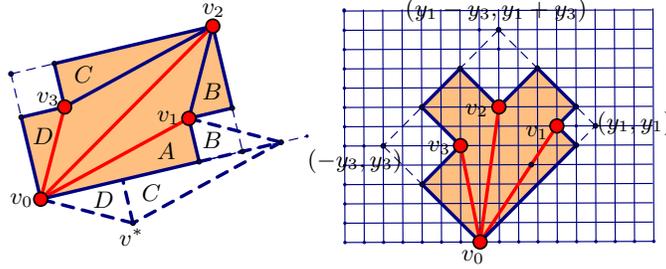


FIGURE 6. Special Cases.

into the parallelogram given by vectors v^* and v_1 . Since v^* is the image of v_3 under a 90° clockwise rotation about the origin, its Λ -coordinates are $(y_3, -x_3)$. Hence the area of this parallelogram is $y_3y_1 + x_1x_3$ and the area of the cap is $x_1y_3 - y_1x_3 + y_3y_1 + x_1x_3$.

Turning to the 45° case, we must have $(x_1 - y_1 + x_2 + y_2 - x_3 + y_3) = (x_1 + y_1 - x_2 + y_2 - x_3 - y_3)$. This gives $x_2 = y_1 - y_3$. Substituting $y_1 - y_3$ for x_2 in the rim coordinates shows that in this case the rim coordinates are $(x_1 + y_2 - x_3, x_1 + y_2 - x_3)$. Turning to area, we easily see that the corners of the rectangle have Λ -coordinates (y_1, y_1) , $(y_1 - y_3, y_1 + y_3)$ and $(-y_3, y_3)$ (in counterclockwise order). Hence the entire rectangle has area $2y_1y_3$ and we need only subtract off the areas of the three corner squares. The area of a square with 45° angled sides is simply the square of its height or width divided by 2. Hence the area of the cap is $2y_1y_3 - \frac{1}{2}((y_1 - x_1)^2 + (y_1 + y_3 - y_2)^2 + (y_3 + x_3)^2)$. For easy reference, we summarize all of these geometric results in the next lemma.

Lemma 8. *Let \mathcal{P} the cap of a set of four isolated degree-3 vertices in the SSG Γ . Assume that the degree-3 vertices are labeled v_0, v_1, v_2 and v_3 and are arranged in counterclockwise order with v_0 on the rim. Assume that the convex hull of these vertices is embedded on Λ with v_0 at the origin and v_1 in the first quadrant. Finally, let v_4 denote the point on the bottom rim nearest to v_0 and let v_i have Λ -coordinates (x_i, y_i) for $i = 1, \dots, 4$. Then:*

- (i) *The rim coordinates are*
 - (a) *If this is a parallelogram cap, $x_2 = x_1 + x_3$, $y_2 = y_1 + y_3$ and the rim coordinates are $(2x_1 + 2y_3, 2y_1 - 2x_3)$.*
 - (b) *If this is a 45° cap, $x_2 = y_1 - y_3$ and the rim coordinates are $(x_1 + y_2 - x_3, x_1 + y_2 - x_3)$.*
- (ii) *If $v' = (x', y')$ and $v'' = (x'', y'')$ are vectors, $x'y'' - x''y' > 0$ when v' and v'' are in counterclockwise order around the origin*

and $x'y'' - x''y' < 0$ when v' and v'' are in clockwise order. In either case $|x'y'' - x''y'|$ is the area of the parallelogram with vertices v_0, v', v'' and $v' + v''$. Thus the area of the cylinder is $|x_4(x_1 - y_1 + x_2 + y_2 - x_3 + y_3) - y_4(x_1 + y_1 - x_2 + y_2 - x_3 - y_3)|$

(a) If this is a parallelogram cap, the area of the cap is

$$x_1y_3 - x_3y_1 + y_1y_3 + x_1x_3 \text{ and the area of the cylinder is } 2|x_4(y_1 - x_3) - y_4(x_1 + y_3)|.$$

(b) If this is a 45° cap, the area of the cap is

$$2y_1y_3 - \frac{1}{2}((y_1 - x_1)^2 + (y_3 + x_3)^2 + (y_1 + y_3 - y_2)^2) \text{ and the area of the cylinder is } |(x_4 - y_4)(x_1 + y_2 - x_3)|.$$

4. THE SYMMETRIES AND SYMMETRY GROUPS OF SSGs

The main result of this section is a complete listing of the symmetries and symmetry groups of the SSGs. A similar study was carried out for general self-dual plane graphs by B. Servatius and H. Servatius [7]. Any plane graph may be embedded on the sphere so that the symmetries of the graph are isometries of the sphere and its symmetry group is a discrete group of isometries on the sphere (See Lemma 1 in [7]). In our investigation of the symmetries and symmetry groups of SSGs, we will use terminology of the isometries of the sphere as in Coxeter and Moser [3]. It is also the case that each symmetry of an SSG must permute the eight degree-3 vertices and therefore may be identified with a subgroup of the permutation group of these vertices. We will also use the terminology of this permutation group. Specifically we will talk about the *stabilizers* of the degree-3 vertices and the *orbits of a symmetry or of a symmetry group* meaning the orbits of the permutation or permutation group of the degree-3 vertices. We will frequently use the result that the product of the size of the orbit containing a degree-3 vertex and the size of its stabilizer equals the order of the symmetry group.

Let $\Gamma = (V, E, F)$ be an SSG. Any symmetry γ of Γ must respect the bipartition $(V_b \cup V_r, V_w)$ of its vertex set. Since all of the degree-3 vertices are in $V_b \cup V_r$, $\gamma(V_w) = V_w$. Any symmetry γ must also respect the bipartition of the auxiliary graph and either preserve the subgraphs $(V_b \cup V_w, E_b)$ and $(V_r \cup V_w, E_r)$ [$\gamma(V_b) = V_b$ and $\gamma(V_r) = V_r$, *color-preserving*] or interchange them [$\gamma(V_b) = V_r$ and $\gamma(V_r) = V_b$, *color-reversing*]. The definition of an SSG requires that Γ admit at least one color-reversing symmetry. The symmetries of Γ may also be classified as *direct (orientation preserving)* or *opposite (orientation reversing)* depending on whether their extension to an isometry of the sphere is direct or opposite. The direct symmetries map segments with SSG-coordinates (p, q) onto segments with SSG-coordinates (p, q)

while opposite symmetries map segments with SSG-coordinates (p, q) onto segments with SSG-coordinates (q, p) .

Lemma 9. *Let $\Gamma = (V_b \cup V_r \cup V_w, E, F)$ be an SSG. Let T_b and T_r denote the degree-3 black and red vertices, respectively. Then any symmetry of Γ must be one of the following:*

- *The direct, color-preserving symmetries:*
 - (i) *rotations of order 4 about antipodal degree-4 vertices from $V_b \cup V_r$;*
 - (ii) *rotations of order 3 about antipodal degree-3 vertices $c_b \in T_b$ and $c_r \in T_r$;*
 - (iii) *rotations of order 2 about antipodal vertices.*
- *The direct, color-reversing symmetries:*
 - (iv) *rotations of order 4 about antipodal vertices from V_w ;*
 - (v) *rotations of order 2 about the centers of antipodal faces.*
- *The opposite, color-preserving symmetries:*
 - (vi) *A reflection through a color-preserving equator*;*
 - (vii) *A rotatory-reflection of order 4 with centers of the same color;*
 - (viii) *The antipodal map α .*
- *The opposite, color-reversing symmetries:*
 - (ix) *A reflection with a color-reversing equator*;*
 - (x) *A rotatory-reflection of order 8 with antipodal degree-3 centers $c_b \in T_b$ and $c_r \in T_r$ and a color-reversing equator*;*
 - (xi) *A rotatory-reflection of order 6 with antipodal degree-3 centers $c_b \in T_b$ and $c_r \in T_r$ and a color-reversing equator*;*
 - (xii) *A rotatory-reflection of order 4 with centers from V_w or one red and one black center and a color-reversing equator*;*
 - (xiii) *The antipodal map α .*

*Color-preserving and color-reversing equators are listed in Figure 7.

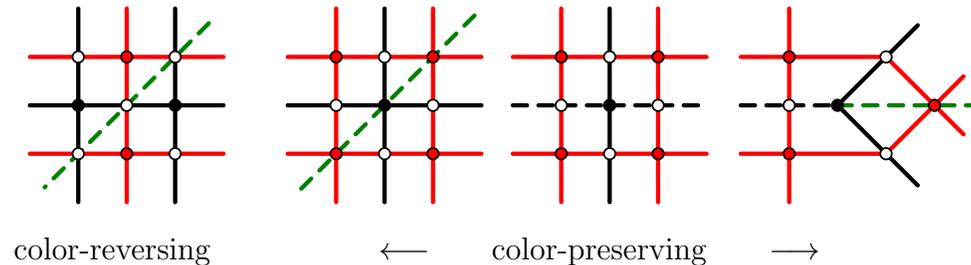


FIGURE 7. Possible equators for a reflection.

Proof. Since Γ is a plane graph any symmetry of Γ corresponds to an isometry of the sphere. Hence, the direct symmetries of Γ must be

rotations and have two antipodal fixed points. We call these fixed points the *centers* of the rotation. Since these rotations are also graph symmetries, the only choices for centers are: vertices, centers of faces and centers of edges. It follows that the only possible orders for a rotation are 2, 3 or 4 (for vertices), 2 or 4 (for faces) and 2 (for edges). Since a rotation of order 2 about the center of an edge interchanges a white vertex with a red or black vertex, it cannot extend to a symmetry of Γ . Since a rotation of order 4 about the center of a face maps the white vertices bounding the face onto the black and red vertices bounding that face, it also cannot extend to a symmetry of Γ . Hence the only possibilities are rotations of order 2, 3 or 4 centered at vertices or half-turns (order 2) centered at faces.

A rotation of order 4 about a red or black vertex is color-preserving while a rotation of order 4 about a white vertex is color-reversing. Hence, rotations of order 4 have both centers in V_w [item (iv) above] or both centers in $V_b \cup V_r$ [item (i)]. The two centers of a rotation of order 3 can only be vertices of degree-3. The remaining six vertices of degree-3 must fall into two orbits of length 3 each. Since there are 4 red and 4 black vertices of degree-3, there must one red and one black center and one red and one black orbit of length 3 [item (ii)]. A rotation of order 2 about the center of a face interchanges the white vertices bounding the face and interchanges the black and red vertices bounding that face; so it could only extend to a color-reversing symmetry of Γ [item (v)]. A rotations of order 2 about a vertex of any color is color-preserving [item (iii)].

The only possible opposite symmetries are reflections and rotatory-reflections. Reflections of the sphere have a plane of reflection which intersects the sphere in a great circle or *equator* that is fixed. If ρ is a reflection of Γ , the equator of its extension to the sphere can intersect a face of Γ only in an edge (in which case it interchanges the faces containing that edge) or a diagonal of the face (in which case it maps that face onto itself). If the equator of a reflection coincides with the diagonal of a face f joining two white vertices v and w , then that equator must include the white-white diagonals of the faces containing v and w but not adjacent to f . It follows that, if an equator contains one white-white diagonal it must be an entire circuit of white-white diagonals. In this case, the reflection is color-reversing [item (ix)]. We will refer to such a circuit of white-white diagonals as a *color-reversing equator*. See the left-hand diagram in Figure 7.

If the equator of a reflection coincides with the diagonal of a face f joining red and black vertices v and w of degree 4, then that equator must include the red-black diagonals of the faces containing v and w

but not adjacent to f . See Figure 7. Now assume that the equator of a reflection coincides with the black (red) edge joining the white vertex v and the black (red) vertex w . Then that equator must include the other black (red) edge containing v . When w has degree 4, the equator must include the opposite black (red) edge containing w ; when w has degree-3, the equator must include the red-black diagonal of the face containing the other two edges at w . In all of these cases, the reflection is color-preserving [item (vi)]. We conclude that a *color-preserving equator* is a circle of edges and red-black diagonals as pictured in the right three diagrams of Figure 7.

Each rotatory-reflection γ in the symmetry group of Γ is the composition of a rotation σ and a reflection ρ where the plane of its equator is perpendicular to the axis of σ . If both σ and ρ are themselves symmetries of Γ , the center and equator of γ are well defined. However, $\gamma = \sigma\rho$ may be a symmetry of Γ even when σ and ρ are not. Since the axis of σ is perpendicular to the plane of ρ , σ and ρ commute. Thus, $\gamma = \sigma\rho = \rho\sigma$ and $\gamma^2 = \sigma^2$. Hence if σ has order greater than 2, we may still recover the center and equator for γ : the centers of γ are the centers of the rotation σ^2 and the equator of γ is the perpendicular plane bisecting the axis of σ . In the case that σ is a half-turn, σ^2 and hence γ^2 is the identity map and there is no uniquely defined axis nor uniquely defined equator. In this case, γ is the unique *antipodal symmetry* and is denoted by α . For each vertex, edge or face X , $\alpha(X)$ is called the *antipodal* vertex, edge or face of X ; the antipodal map α may be either color-preserving [item (viii)] or color-reversing [item (xiii)].

Suppose that γ is a rotatory-reflection of order $n > 2$. Then γ^2 is a rotation of order $\frac{n}{2}$. Hence $n = 4, 6$ or 8 . If n is 8 then the vertices of degree-3 must form an orbit of length 8. Since there are 4 red and 4 black vertices of degree-3, γ must be color-reversing. However, γ^2 is a color-preserving rotation of order 4. Hence, the centers of γ must be in $V_b \cup V_r$, and since γ interchanges the centers, one is red and the other is black [item (x)].

Using the same reasoning, we conclude that a rotatory-reflection γ of order 6 is color-reversing and has an orbit consisting of three red and three black degree-3 vertices. The remaining two vertices of degree-3 must be the degree-3 red and the degree-3 black centers of the rotation γ^2 [item (xi)].

Finally, the square of a rotatory-reflection of order 4 is a color-preserving half-turn; hence its centers are vertices. Therefore, a color-preserving rotatory-reflection of order 4 must have both centers of the same color [item (vii)] and color-reversing rotatory-reflection of order

4 must have one red center and one black center or both centers white [item (xii)]. \square

Since the composition of two color-preserving symmetries is also color-preserving, the color-preserving symmetries form a subgroup of the symmetry group of the SSG. Since the composition of two color-reversing symmetries is a color-preserving symmetry, the color-preserving symmetries form an index 2 subgroup. This subgroup may be identified with the symmetry group of the subgraph $(V_b \cup V_w, E_b)$ and the underlying self-dual spherical grid Γ_b . So, in Theorem 1 below, we list a pair: the symmetry group of the SSG, Γ , and the symmetry group of the underlying self-dual spherical grid, Γ_b . This result is similar to Theorem 1 in Servatius and Servatius [7]. The main difference is that here the groups are specifically tied to spherical grids and then to the templates. A few of our groups are consolidated into a single class in [7].

Theorem 1. *Following is a complete list of all possible symmetry groups \mathcal{A}_Γ of an SSG Γ along with the induced symmetry group \mathcal{A}_Δ of Δ , the underlying self-dual SG.*

- *Order 2*
 - (i) $\mathcal{A}_\Gamma = \{\text{one color-reversing reflection } \rho, \iota\}; \mathcal{A}_\Delta = \{\iota\}.$
 - (ii) $\mathcal{A}_\Gamma = \{\text{the color-reversing antipodal map } \alpha, \iota\}; \mathcal{A}_\Delta = \{\iota\}.$
 - (iii) $\mathcal{A}_\Gamma = \{\text{one color-reversing half-turn } \tau, \iota\}; \mathcal{A}_\Delta = \{\iota\}.$
- *Order 4*
 - (iv) $\mathcal{A}_\Gamma = \{\text{color-reversing reflections, } \rho_1 \text{ and } \rho_2, \text{ a color-preserving half-turn } \tau = \rho_1\rho_2 \text{ and } \iota\}. \mathcal{A}_\Delta = \{\tau, \iota\}.$
 - (v) $\mathcal{A}_\Gamma = \{\text{One color-reversing reflection } \rho, \text{ one color-preserving reflection } \eta, \text{ the color-reversing half-turn } \tau = \rho\eta \text{ and } \iota\}. \mathcal{A}_\Delta = \{\eta, \iota\}.$
 - (vi) $\mathcal{A}_\Gamma = \{\text{One color-reversing reflection } \rho, \text{ one color-preserving half-turn } \tau, \text{ the color-reversing antipodal map } \alpha = \rho\tau \text{ and } \iota\}. \mathcal{A}_\Delta = \{\tau, \iota\}.$
 - (vii) $\mathcal{A}_\Gamma = \{\text{One color-reversing reflection } \rho, \text{ one color-reversing half-turn } \tau, \text{ the color-preserving antipodal map } \alpha = \rho\tau \text{ and } \iota\}. \mathcal{A}_\Delta = \{\alpha, \iota\}.$
 - (viii) $\mathcal{A}_\Gamma = \{\text{One color-preserving reflection } \eta, \text{ one color-reversing half-turn } \tau, \text{ the color-reversing antipodal map } \alpha = \eta\tau \text{ and } \iota\}. \mathcal{A}_\Delta = \{\eta, \iota\}.$
 - (ix) $\mathcal{A}_\Gamma = \{\text{One color-reversing rotatory-reflection } \phi \text{ of order 4 and its powers}\}. \mathcal{A}_\Delta = \{\phi^2, \iota\}.$

- (x) $\mathcal{A}_\Gamma = \{ \text{Three half-turns with pairwise orthogonal axes: two color-reversing, } \tau_1, \tau_2 \text{ and one color-preserving, } \tau_3 = \tau_1\tau_2 \}. \mathcal{A}_\Delta = \{ \tau_3, \iota \}.$
- Order 6
 - (xi) $\mathcal{A}_\Gamma = \{ \text{One color-preserving rotation } \sigma \text{ of order 3 and its powers and three color-reversing half-turns } \tau_1, \tau_2, \tau_3 \}. \mathcal{A}_\Delta = \{ \sigma, \sigma^2, \iota \}.$
- Order 8
 - (xii) $\mathcal{A}_\Gamma = \{ \text{One color-preserving rotation } \sigma \text{ of order 4 and its powers and } \tau_1, \tau_2, \tau_3, \tau_4, \text{ four color-reversing half-turns } \}. \mathcal{A}_\Delta = \{ \sigma, \sigma^2\sigma^3, \iota \}.$
 - (xiii) $\mathcal{A}_\Gamma = \{ \text{One color-reversing rotation } \sigma \text{ of order 4 and its powers, two color-reversing half-turns } \tau_1 \text{ and } \tau_3, \text{ two color-preserving half-turns } \tau_2 \text{ and } \tau_4 \}. \mathcal{A}_\Delta = \{ \sigma^2, \tau_2, \tau_4, \iota \}.$
 - (xiv) $\mathcal{A}_\Gamma = \{ \text{3 color-reversing reflections } \rho_1, \rho_2 \text{ and } \rho_3, \text{ the (color-reversing) antipodal map } \alpha = \rho_1\rho_2\rho_3, \text{ three color-preserving half-turns } \tau_1 = \rho_1\rho_2, \tau_2 = \rho_1\rho_3, \tau_3 = \rho_2\rho_3, \text{ and the identity.} \}. \mathcal{A}_\Delta = \{ \tau_1, \tau_2, \tau_3, \iota \}.$
 - (xv) $\mathcal{A}_\Gamma = \{ \text{2 color-reversing reflections, } \rho_1, \rho_2, \text{ and one color-preserving reflection, } \eta, \text{ the (color-preserving) antipodal map } \alpha = \rho_1\rho_2\eta, \text{ two color-reversing half-turns } \tau_1 = \rho_1\eta, \tau_2 = \rho_2\eta, \text{ one color-preserving half-turn } \tau = \rho_1\rho_2, \text{ and the identity.} \}. \mathcal{A}_\Delta = \{ \eta, \alpha, \tau, \iota \}.$
 - (xvi) $\mathcal{A}_\Gamma = \{ \text{2 color-preserving reflections } \rho_1 \text{ and } \rho_2, \text{ one color-reversing reflection, } \rho_3, \text{ the color-reversing antipodal map } \alpha = \rho_1\rho_2\rho_3, \text{ two color-reversing half-turns } \tau_1 = \rho_2\rho_3, \tau_2 = \rho_1\rho_3, \text{ one color-preserving half-turn } \tau_3 = \rho_2\rho_3, \text{ and the identity.} \}. \mathcal{A}_\Delta = \{ \tau_3, \rho_1, \rho_2, \iota \}.$
 - (xvii) $\mathcal{A}_\Gamma = \{ \text{a color-preserving rotatory-reflection } \sigma \text{ of order 4 and its powers, two color-reversing half-turns } \tau_1 \text{ and } \tau_2 \text{ and two color-reversing reflections } \rho_1 \text{ and } \rho_2 \text{ where } \tau_1\tau_2 = \rho_1\rho_2 = \sigma^2. \mathcal{A}_\Delta = \{ \iota, \sigma, \sigma^2, \sigma^3 \}.$
 - (xviii) $\mathcal{A}_\Gamma = \{ \text{a color-reversing rotatory-reflection } \sigma \text{ of order 4 and its powers, two color-preserving half-turns } \tau_1 \text{ and } \tau_2 \text{ and two color-reversing reflections } \rho_1 \text{ and } \rho_2 \text{ where } \tau_1\tau_2 = \rho_1\rho_2 = \sigma^2. \mathcal{A}_\Delta = \{ \iota, \sigma^2, \tau_1, \tau_2 \}.$
 - (xix) $\mathcal{A}_\Gamma = \{ \text{a color-reversing rotatory-reflection } \sigma \text{ of order 4 and its powers, two color-reversing half-turns } \tau_1 \text{ and } \tau_2 \text{ and two color-preserving reflections } \rho_1 \text{ and } \rho_2 \text{ where } \tau_1\tau_2 = \rho_1\rho_2 = \sigma^2. \mathcal{A}_\Delta = \{ \iota, \sigma^2, \rho_1, \rho_2 \}.$
- Order 12

- (xx) $\mathcal{A}_\Gamma = \{ \text{One color-preserving rotation } \sigma \text{ of order 3 and its powers, three color-reversing half-turns, } \tau_1, \tau_2, \tau_3, \text{ three color-preserving reflections } \rho_1, \rho_2, \rho_3 \text{ through planes containing the axis of } \sigma, \text{ one color-reversing reflection } \rho_0 \text{ through the plane perpendicular to the axis of } \sigma \text{ plus } \sigma\rho_0 \text{ and } \sigma^2\rho_0 \}. \mathcal{A}_\Delta = \{ \rho_1, \rho_2, \rho_3, \sigma, \sigma^2, \iota \}.$
- (xxi) $\mathcal{A}_\Gamma = \{ \text{One color-reversing rotatory-reflection } \phi \text{ of order 6 and its powers, three color-reversing half-turns, three color-preserving reflections } \rho_1, \rho_2, \rho_3 \text{ through planes containing the axis of } \phi \}. \mathcal{A}_\Delta = \{ \rho_1, \rho_2, \rho_3, \phi^2, \phi^4, \iota \}.$
- Order 16
- (xxii) $\mathcal{A}_\Gamma = \{ \text{One color-reversing rotatory-reflection } \phi \text{ of order 8 and its powers, four color-reversing half-turns, four color-preserving reflections } \rho_1, \rho_2, \rho_3, \rho_4 \text{ through planes containing the axis of } \phi \}. \mathcal{A}_\Delta = \{ \rho_1, \rho_2, \rho_3, \rho_4, \phi^2, \phi^4, \phi^6, \iota \}.$
- (xxiii) $\mathcal{A}_\Gamma = \{ \text{One color-preserving rotation } \sigma \text{ of order 4 and its powers, four color-reversing half-turns, four color-preserving reflections, } \rho_1, \rho_2, \rho_3, \rho_4, \text{ through planes containing the axis of } \sigma, \text{ one color-reversing reflection } \rho \text{ through the plane perpendicular to the axis of } \sigma \text{ plus the rotatory-reflections } \sigma\rho, \sigma^2\rho \text{ and } \sigma^3\rho \}. \mathcal{A}_\Delta = \{ \rho_1, \rho_2, \rho_3, \rho_4, \sigma, \sigma^2, \sigma^3, \iota \}.$
- (xxiv) $\mathcal{A}_\Gamma = \{ \text{One color-reversing rotation } \sigma \text{ of order 4 and its powers, two color-reversing half-turns } \tau_1 \text{ and } \tau_3, \text{ two color-preserving half-turns } \tau_2 \text{ and } \tau_4, \text{ two color-reversing reflections } \rho_1 \text{ and } \rho_3, \text{ color-preserving reflections } \rho_2 \text{ and } \rho_4 \text{ all through planes containing the axis of } \sigma, \text{ one color-preserving reflection } \rho \text{ with equator perpendicular to the axis of } \sigma \text{ and its compositions with the powers of } \sigma \}. \mathcal{A}_\Delta = \{ \tau_2, \tau_4, \rho_2, \rho_4, \rho, \sigma^2, \alpha = \sigma^2\rho, \iota \}.$
- (xxv) $\mathcal{A}_\Gamma = \{ \text{One color-reversing rotation } \sigma \text{ of order 4 and its powers, two color-reversing half-turns } \tau_1 \text{ and } \tau_3, \text{ two color-preserving half-turns } \tau_2 \text{ and } \tau_4, \text{ two color-reversing reflections } \rho_1 \text{ and } \rho_3, \text{ color-preserving reflections } \rho_2 \text{ and } \rho_4 \text{ all through planes containing the axis of } \sigma, \text{ one color-reversing reflection } \rho \text{ with equator perpendicular to the axis of } \sigma \text{ and its compositions with the powers of } \sigma \}. \mathcal{A}_\Delta = \{ \tau_2, \tau_4, \rho_2, \rho_4, \sigma\rho, \sigma^2, \sigma^3\rho, \iota \}.$
- Order 24
- (xxvi) $\mathcal{A}_\Gamma = \text{the direct symmetry group of the cube and } \mathcal{A}_\Delta = \text{the direct symmetry group of the tetrahedra.}$
- Order 48

(xxvii) $\mathcal{A}_\Gamma =$ the full symmetry group of the cube and $\mathcal{A}_\Delta =$ the full symmetry group of the tetrahedra.

Proof. We will actually prove only half of the theorem here. Namely that this is a complete list of *possibilities*. The fact that each group does occur will be established in the last section where the smallest example of each type will be constructed.

We start by noting that either all of the symmetries of an SSG are direct or the direct symmetries form an index-2 subgroup. We also note that the composition of two half-turns is a rotation by twice the angle between their axes about an axis perpendicular to the plane containing their axes. Hence, if the SSG admits no rotations of order 3 or 4, its direct subgroup is one of the following:

- (a) the identity, $\{\iota\}$;
- (b) one half-turn and the identity, $\{\iota, \tau\}$;
- (c) three half-turns with pairwise orthogonal axes and the identity, $\{\iota, \tau_1, \tau_2, \tau_3\}$.

It follows from Lemma 9 that the only symmetries that fix degree-3 vertices are color-preserving reflections and color-preserving rotations of order 3. Hence the only possible non-trivial stabilizers for degree-3 vertices are:

- (1) a color-preserving reflection (order 2),
- (2) a rotation of order 3 (order 3) or
- (3) a rotation of order 3 and three color-preserving reflections (order 6).

Let σ denote a counter clockwise rotation of order 3 or 4 for the SSG, let c and c' denote the centers of σ . Let v_0 be a degree-3 vertex nearest to c but not equal to c and let $v_i = \sigma^i(v_0)$. Let \mathcal{C} denote the cap determined by c, v_0, v_1 and v_2 (if σ has order 3) or by v_0, v_1, v_2 and v_3 (if σ has order 4). If σ has order 3, we choose the Λ -coordinate system centered at v_0 with the center c in the first quadrant having Λ -coordinates (x, y) . Then, inserting a 90° wedge at c , we have $v_2 = (2x, 2y)$ and $v_3 = (x - y, x + y)$. If σ has order 4, let (x, y) be the Λ -coordinates of v_1 . Then $v_2 = (x - y, x + y)$ and $v_3 = (-y, x)$. By Lemma 8, the rim coordinates of \mathcal{C} are $3(x + y, -x + y)$ and $4(x, y)$, respectively. We may carry out the same analysis for the other cap with v'_0 at the origin and (x', y') as the Λ -coordinates of c' or v'_1 in the bottom cap. Hence the rim coordinates of the bottom cap are $3(x' + y', -x' + y')$ and $4(x', y')$, respectively. However these rims have the same coordinates. Hence $x' = x$ and $y' = y$. We conclude that the two caps are isomorphic.

Now suppose that an SSG has been decomposed into two caps and a cylinder and assume that the two caps are congruent under a direct isometry. Relabeling if needed, we may assume that v'_0 is the degree-3 vertex on the bottom rim that is the image of v_0 under this direct isometry. The segment joining v_0 to v'_0 splits the cylinder into a parallelogram and the half-turn about the midpoint of this segment interchanges the caps and the vertices v_0 and v'_0 . Hence this half-turn is a symmetry of the SSG.

If the SSG admits a rotation of order 3 and a rotation of order 4 or two rotations of order 4 with different axes, then it is easy to see that all eight degree-3 vertices lie on the same orbit. If there are no reflections the vertex stabilizers have order 3, the group has order 24 and must be the direct symmetry group of the cube. In this case, it is natural to visualize the SSG as the cube with the eight degree-3 vertices as corners. It follows from this and the previous argument that remaining possible direct subgroups are:

- (d) a rotation, σ of order 3 and its powers and three half-turns with axes perpendicular to the axis of σ , $\{\iota, \sigma, \sigma^2, \tau_1, \tau_2, \tau_3\}$;
- (e) a rotation, σ of order 4 and its powers and four half-turns with axes perpendicular to the axis of σ , $\{\iota, \sigma, \sigma^2, \sigma^3, \tau_1, \tau_2, \tau_3, \tau_4\}$;
- (f) the 24 element direct symmetry group of the cube.

With these background results, we proceed with the identification of all possible SSG symmetry groups. For a symmetry group of order 2, we need a color-reversing symmetry of order 2. The only such symmetries listed in Lemma 9 are a color-reversing reflection, a color-reversing half-turn and the color-reversing antipodal map. This accounts for items (i), (ii), the only groups with direct subgroup (a), and (iii), only group equal to (b).

We now turn to the groups of order 4. The only direct group of order 4 is (c) above and leads directly to group (x). The remaining groups of order 4 are obtained by including two opposite symmetries with the direct subgroup (b) and the composition of these two opposite symmetries must be the half-turn. Hence the only options for a group of order 4 are:

- (1) one half-turn τ and two reflections through orthogonal planes containing the axis of τ and the identity;
- (2) one half-turn τ , one reflection through the plane orthogonal to the axis of τ , the antipodal map α and the identity;
- (3) a rotatory-reflection of order 4 and its powers;

In all cases, two of the symmetries must be color-reversing. We consider these possibilities one at a time. There are two ways to select the

two symmetries to be color-reversing from the half-turn τ and the two reflections through orthogonal planes containing the axis of τ . These are listed as items (iv) and (v). There are three ways to select the two symmetries to be color-reversing from the half-turn τ , the reflection through the plane orthogonal to the axis of τ and the antipodal map α . These are listed as items (vi), (vii) and (viii). A rotatory-reflection of order 4 must be color-reversing and results in item (ix).

A symmetry group of order 6 must be group (d) above: if it included opposite symmetries, the index 2 direct subgroup would have order 3 and that is not possible. So the group is $\{\iota, \sigma, \sigma^2, \tau_1, \tau_2, \tau_3\}$. Since $\sigma^3 = \iota$, σ must be color-preserving and so τ_1, τ_2, τ_3 must be color-reversing: group (xi). Moving on to order 8, either the group equals (e) above or has (c) as its direct subgroup. If it equals (e) either the rotation of order 4 is color-preserving, leading to group (xii), or it is color-reversing, leading to group (xiii). Adding four opposite symmetries to (c) can be done in only two ways:

- (1) three reflections with pairwise orthogonal planes of reflection intersecting in the axes of the half-turns and the antipodal map.
- (2) the odd powers of a rotatory-reflection σ of order 4 and two reflections with planes of reflection containing the axis of σ .

Three distinct versions of the first group listed can occur. Since the reflections generate the group, at least one of them is color-reversing giving the options of one, two or three color-reversing reflections: items (xiv), (xv) and (xvi). There are also three distinct versions of the second group. In this case, the group consists of ι, σ, σ^2 , and σ^3 , half-turns τ_1 and τ_2 distinct from the half-turn σ^2 , and reflections ρ_1 and ρ_2 . Since σ^2 is color-preserving and $\tau_1\tau_2 = \sigma^2 = \rho_1\rho_2$, our options for four color-reversing symmetries are: (xvii) τ_1, τ_2, ρ_1 , and ρ_2 ; (xviii) σ, σ^3, ρ_1 , and ρ_2 ; (xix) σ, σ^3, τ_1 , and τ_2 .

There is no direct group of order 12; hence any group of order 12 must have (d) as its direct subgroup $\{\iota, \sigma, \sigma^2, \tau_1, \tau_2, \tau_3\}$. We note that since $\sigma^3 = \iota$, σ must be color-preserving and since the half-turns interchange the (differently colored) centers of σ they must all be color-reversing. To get the full group we simply add one opposite symmetry and compute its compositions with the symmetries in (d). It is easy to see that at least one of the six added opposite symmetries must be a reflection through a plane containing the axis of σ . If this plane also contains the axis of one of the half-turns, we get group (xx); if this plane bisects the angle between consecutive half-turn axes, we get group (xxi).

Similarly, there is no direct group of order 16 and all symmetry groups of order 16 must contain (e) as direct subgroup. Again there are just two ways to add the opposite symmetries: one option results in a rotatory-reflection of order 8, the other is a reflection through a plane perpendicular the axis of the rotation of order 4. In the first case, the rotatory-reflection having an orbit of length 8 must be color-reversing and group (xxii) is the result. In the second case there are three different ways to choose the color-preserving subgroup: (xxiii), (xxiv) and (xv).

Finally, as already noted, the only possibilities with more than one rotation of order greater than 2 are the direct and full symmetry groups of the cube - item (xxvi) and (xxvii). \square

5. TEMPLATES FOR SSGs

In this section we describe how to construct every SSG and hence every self-dual spherical grid. Let $\Gamma = (V_b \cup V_r \cup V_w, E_b \cup E_r)$ be an SSG. We organize our construction about the symmetry group of Γ , as listed in Theorem 1. We will use just four templates: Template 1 for all groups containing three half-turns with pairwise orthogonal axes $\{(x), (xii), (xiii), (xiv), (xv), (xvi), (xvii), (xviii), (xix), (xxii), (xxiii), (xxiv), (xxv), (xxvi), (xxvii)\}$; Template 2 for the three groups that include three half-turns with axes in the same plane and no additional half-turns $\{(xi), (xx), (xxi)\}$; Template 3 for the groups that include exactly one half-turn $\{(iii), (iv), (v), (vi), (vii), (viii), (ix)\}$; Template 4 for the two groups that include no half-turns $\{(i), (ii)\}$.

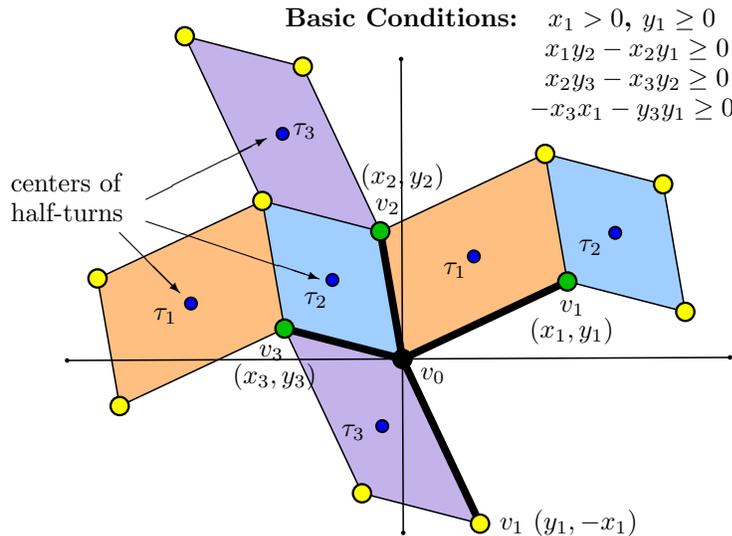
Let Γ be an SSG with a half-turn $\tau \in \mathcal{A}_\Gamma$. Let c and c' denote the centers of τ . Let v_1 be a degree-3 vertex closest to c and let $v_3 = \tau(v_1)$. Let v_0 be a degree-3 vertex next closest to c after v_1 and v_3 . Then $v_0, v_1, v_2 = \tau(v_0)$ and v_3 form the vertices of a parallelogram \mathcal{P}_τ . We call \mathcal{P}_τ “the” *parallelogram of τ at c* . It is possible that there could be several degree-3 vertices equally close to c at either step of the selection process and a different choice may yield a different parallelogram. Nevertheless no matter how we choose the vertices of \mathcal{P}_τ , the remaining equally close vertices lie outside the cap of \mathcal{P}_τ . To see this, we observe that the entire cap lies in a rectangle that is inscribed in the circle with center c passing through v_0 and v_2 . The remaining degree-3 vertices must lie on or outside of that circle and hence outside the cap of \mathcal{P}_τ . The remaining four degree-3 vertices must then be the vertices of the *parallelogram of τ at c'* , denoted by \mathcal{P}'_τ .

Lemma 10. *Let Γ be an SSG with a half-turn $\tau \in \mathcal{A}_\Gamma$. Then \mathcal{P}_τ and \mathcal{P}'_τ are directly congruent (congruent under a direct isometry of the*

sphere) if and only if there exists another half-turn $\tau^* \in \mathcal{A}_\Gamma$ with axis perpendicular to the axis of τ .

Proof. Let Γ be an SSG with a half-turn $\tau \in \mathcal{A}_\Gamma$. If a half-turn τ^* exists with axis perpendicular to the axis of τ , it must interchange \mathcal{P}_τ and \mathcal{P}'_τ . Now suppose that \mathcal{P}_τ and \mathcal{P}'_τ are directly congruent and consider the cap and cylinder decomposition of Γ . Let v_0 and v_2 denote the vertices of \mathcal{P}_τ on the top rim and v'_0 and v'_2 denote the vertices of \mathcal{P}'_τ on the bottom rim. One easily checks that the segments joining v_0 to v'_0 and v_2 to v'_2 partition the cylinder into two congruent parallelograms and that the half-turn τ^* about their centers interchanges the caps and is a symmetry of Γ . \square

It follows from Lemma 10 that given an SSG with three half-turns τ_1 , τ_2 and τ_3 with perpendicular axes, \mathcal{P}_{τ_i} and \mathcal{P}'_{τ_i} are congruent for each index i . Since for each index i the vertices of \mathcal{P}_{τ_i} and \mathcal{P}'_{τ_i} account for all eight degree-3 vertices, each degree-3 vertex is the vertex of three different parallelograms as pictured in **Template 1**.



The parallelograms of the half-turns fit together to form a paralleloiped with the degree-3 vertices as corners. We select for the origin v_0 , one of the degree-3 vertices, and one of its neighbors v_1 to be in the first quadrant; then label the remaining neighbors v_2 and v_3 in counterclockwise order. The Basic Conditions ensure this orientation: $x_1 y_2 - x_2 y_1 \geq 0$ implies that v_2 is counterclockwise from v_1 ; $x_2 y_3 - x_3 y_2 \geq 0$ implies that v_3 is counterclockwise from v_2 ; $-x_3 x_1 - y_3 y_1 \geq 0$ implies that v_1 is counterclockwise from v_3 .

Lemma 11. *Let Γ be a SSG where \mathcal{A}_Γ admits three half-turns with pairwise perpendicular axes. Assume that it has been embedded in Λ , as illustrated in **Template 1**. Assume that v_0 , the degree-3 vertex at the origin, is black (hence the axes are black). Let τ_i denote the half-turn corresponding to the parallelogram defined by v_i and v_{i+1} (indices read mod 3). Then:*

- (i) *The number of vertices of the underlying self-dual SG is given by $\frac{x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 - x_3x_1 - y_3y_1}{2} + 1$.*
- (ii) *$x_i \equiv y_i \pmod{2}$ for all i and $x_i \equiv 1 \pmod{2}$ for at least one index.*
- (iii) *If $x_i \equiv x_{i+1} \pmod{2}$ (indices read mod 3), then τ_i is color-preserving, otherwise τ_i is color-reversing.*
- (iv) *If \mathcal{A}_Γ admits an orientation reversing symmetry, Γ may be oriented so that one of the following conditions holds:*
 - (a) *$x_i = y_i$ for all i (all edges of the parallelepiped have SSG-coordinates of the form (s, s));*
 - (b) *$x_1 = y_1$ and $y_2 = y_3 = 0$ (one edge of the parallelepiped has SSG-coordinates of the form (s, s) , the remaining two have SSG-coordinates of the form (s));*
 - (c) *$x_2 = -x_1$, $y_2 = y_1$ and $y_3 = 0$ (one edge of the parallelepiped has SSG-coordinates of the form (s) , the remaining two have SSG-coordinates of the form (s, t) and (t, s));*
- (v) *v_0 is the center of a rotation of order 3 if and only if $x_2 = -y_1$, $y_2 = x_1$, $x_3 = -x_1$ and $y_3 = -y_1$ (all edge of the parallelepiped at v_0 have the same SSG-coordinates).*
- (vi) *If Γ admits a rotation σ of order 4, we assume that the center of τ_1 is actually the center of σ with $\tau_1 = \sigma^2$. In this case, $x_2 = -y_1$ and $y_2 = x_1$. In fact, the parallelograms of τ_1 are squares; all edges of these parallelograms have the same SSG-coordinates (p, q) . Furthermore, if p and q are even σ is color-preserving, if p and q are odd σ is color-reversing. In either case, $\tau_1 = \sigma^2$ is color-preserving.*

Proof.

- (i) $x_1y_2 - x_2y_1$ is the area of the parallelogram that is defined by the vectors (x_1, y_1) and (x_2, y_2) with similar formulas for the areas remaining parallelograms at the origin. Adding these these areas and doubling gives the total area. Then dividing by 4 and adding 1 gives the number of vertices of the underlying self-dual SG by Lemma 7.

- (ii) Since the origin is a black vertex, v_i will be black if and only if $x_i \equiv y_i \equiv 0 \pmod{2}$ and red if and only if $x_i \equiv y_i \equiv 1 \pmod{2}$. If $x_i \equiv y_i \equiv 0 \pmod{2}$ for all i , then all degree-3 vertices would be black.
- (iii) If $x_i \equiv x_{i+1} \pmod{2}$, then v_i and v_{i+1} are the same color and τ_i is color-preserving. If $x_i \not\equiv x_{i+1} \pmod{2}$, then v_i and v_{i+1} are different colors and τ_i is color-reversing.
- (iv) Let $\mathcal{P}_i \cong \mathcal{P}'_i$ denote the parallelograms of τ_i . We note first that under any opposite symmetry \mathcal{P}_1 is mapped onto itself, onto \mathcal{P}'_1 or onto one of the other four parallelograms. It follows that either all parallelograms have reflective symmetry or one pair has reflective symmetry and the other two pairs are reflections of one another. We may assume that \mathcal{P}_1 has reflective symmetry. Thus \mathcal{P}_1 admits two reflections. Either the axes of the reflections of \mathcal{P}_1 are perpendicular to its sides and \mathcal{P}_1 is a rectangle or the axes of the reflections of \mathcal{P}_1 are its diagonals. In the first case the SSG-coordinates of its sides must have the form (p, p) and the axis of reflection must carry on through the sides of the neighboring parallelograms forcing all sides of the parallelepiped to have SSG-coordinates of the form (p, p) ; option (a). There are just two ways that the axes of the reflections of \mathcal{P}_1 can be its diagonals giving options (b) and (c).
- (v) If v_0 is the center of a rotation of order 3 the vector to v_2 is the image of the vector to v_1 under the 90° counterclockwise rotation: hence $(x_2, y_2) = (-y_1, x_1)$; similarly, $(x_3, y_3) = (-x_1, -y_1)$.
- (vi) By theorem 1 if σ has order 4, σ^2 is one of a set of three half-turns with pairwise orthogonal axes. Since the parallelograms of σ^2 are mapped onto themselves under a 90° rotation they are squares and the edges at v_0 are perpendicular. Hence $(x_2, y_2) = (-y_1, x_1)$. Adjacent degree-3 vertices around this square face will have the same color when x_1 and y_1 are even and they will alternate in color when x_1 and y_1 are odd.

□

We now consider each of the symmetry groups described by Template 1.

\mathcal{A}_Γ is of Type (xxvii). Since the vertex stabilizers have order 6, we have a rotation of order 3 about the origin giving $x_2 = -y_1$, $y_2 = x_1$, $x_3 = -x_1$ and $y_3 = -y_1$. We also have a reflection fixing the origin, hence we have just one parameter p : $x_1 = y_1 = y_2 = p$ and $x_2 = x_3 =$

$y_3 = -p$. To avoid all vertices having the same color, p must be a positive odd integer. The number of vertices in Γ_b is $3p^2 + 1$ and taking $p = 1$ gives the smallest example: the tetrahedron.

\mathcal{A}_Γ is of Type (xxvi). Since the vertex stabilizers have order 3, we have a rotation of order 3 about the origin giving $x_2 = -y_1$, $y_2 = x_1$, $x_3 = -x_1$ and $y_3 = -y_1$. Since we also have no orientation reversing symmetries, we have two parameters p and q : $x_1 = y_2 = -x_3 = p$ and $-x_2 = -y_3 = y_1 = q$. Furthermore, p and q must be distinct positive odd integers. We then have that the number of vertices of Γ_b is $\frac{3}{2}(p^2 + q^2) + 1$. Hence taking $p = 1$ and $q = 3$ gives the smallest example with 16 vertices.

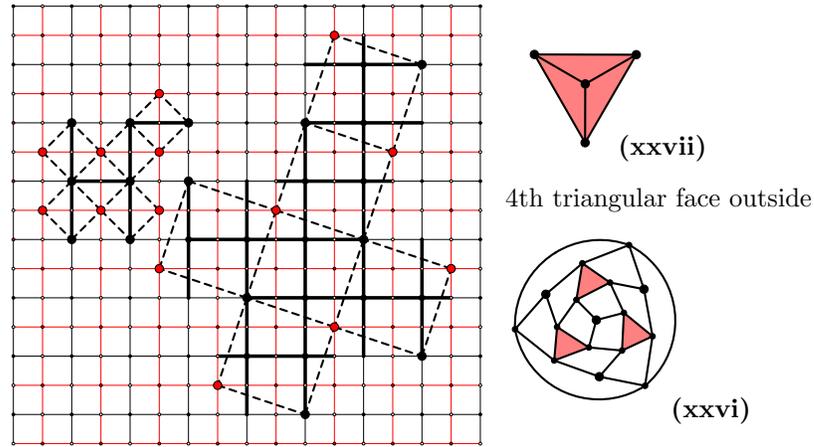


FIGURE 8. Smallest Examples (xxvii) & (xxvi)

For the next three cases, we have a single rotation σ of order 4 with four reflections through planes containing the axis of σ . It follows that the SSG-coordinates of the sides of the square face must be all of the form (p) or all of the form (p, p) . The reflection through the plane containing a degree-3 vertex interchanges the two edges of the square face and fixes the third edge leaving that vertex. We conclude that the three edges at a degree-3 vertex are all of the form (p) or all of the form (p, p) . If they were all of the form (p) all degree-3 vertices would have the same color. Hence, in these cases $x_1 = y_1 = (-x_2) = y_2 = p$ and $x_3 = y_3 = -q$. Parameters p and q must be distinct positive integers to avoid a rotation of order 3, and at least one is odd to ensure that half of the degree-3 vertices are of each color. The number of vertices is $p^2 + 2pq + 1$. Let ρ denote the reflection through the plane perpendicular to the axis σ . Then the options are:

\mathcal{A}_Γ is of Type (xxv). σ and ρ are both color-reversing: p and q odd. To get the smallest example, let $p = 1$ and $q = 3$ giving $|V_b| = 8$.

\mathcal{A}_Γ is of Type (xxiv). σ color-reversing and ρ color-preserving: p odd and q even. To get the smallest example, let $p = 1$ and $q = 2$ giving $|V_b| = 6$.

\mathcal{A}_Γ is of Type (xxiii). σ color-preserving and ρ color-reversing: p even and q odd. To get the smallest example, let $p = 2$ and $q = 1$ giving $|V_b| = 9$.

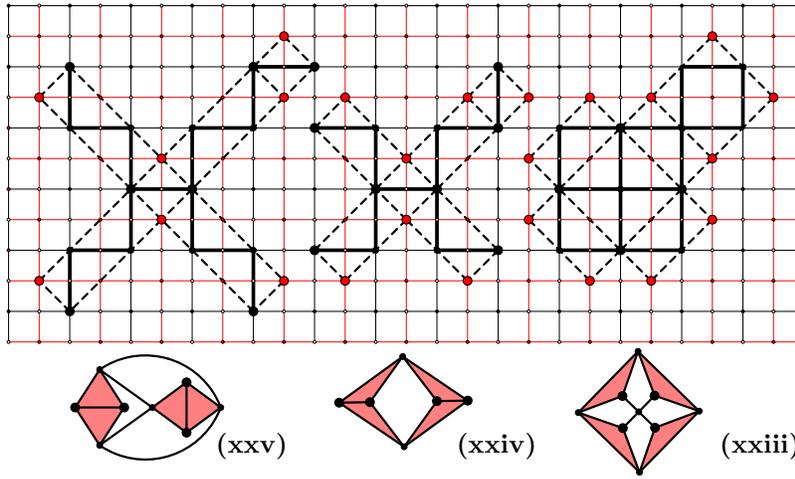


FIGURE 9. Smallest Examples (xxv), (xxiv) & (xxiii)

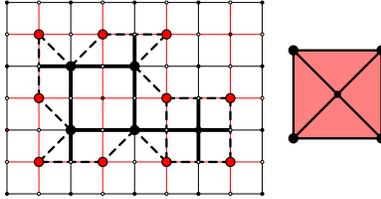


FIGURE 10. Smallest Example (xxii)

\mathcal{A}_Γ is of Type (xxii). Since we have a color-reversing rotatory-reflection ϕ of order 8, we have that the rotation of order 4, ϕ^2 , and the half-turn $\phi^4 = \tau_1$ are both color-preserving. It follows that the τ_1 parallelogram is a square. Since there are also reflections through planes containing the axis of ϕ , the edge SSG-coordinates are of the form (r) or (r, r) where r is even. Now the bottom parallelogram is congruent to the top parallelogram but is rotated 45° . Since the plane of reflection of ϕ is perpendicular to the axis of ϕ , we conclude that v_0 (the origin), v_2 and v_3 form an isosceles triangle with the base vertices

v_0 and v_2 having the same color and v_3 having a different color. This can only happen if $y_2 = x_1 = 2p$, $x_2 = y_1 = 0$ and $x_3 = -q$ and $y_3 = p$ where both p and q are odd. Here we have that the number of vertices of Γ_b is $2p^2 + 2pq + 1$. Hence taking $p = 1$ and $q = 1$ gives the smallest example with 5 vertices.

Cases of Types (xxi) and (xx) are represented by Template #2 and will be considered later. We move down to groups of order 8. The next three cases (xix), (xviii) and (xvii) all have a rotatory-reflection σ of order 4 and we assume that $\sigma^2 = \tau_1$ in the template. They also have reflections through perpendicular planes containing the axis of σ . Thus each the two parallelograms for τ_1 must have two perpendicular axes of symmetry. Furthermore, two parallelograms for τ_1 must be perpendicular to one another when viewed along the axis of τ_1 . There are three different ways to accomplish this:

- (a) Let the axes of symmetry be the diagonals of the parallelogram, that is sides of the parallelogram have SSG-coordinates $(p, q), (q, p), (p, q), (q, p)$ in cyclic order. Specifically: $x_1 = y_2 = p$, $x_2 = y_1 = q$, $x_3 = -2r$, $y_3 = 0$, where $p > q > 0$, $r > 0$ and p, q odd (without this condition all vertices would be the same color). We have $|V_b| = \frac{p^2 - q^2}{2} + 2pr + 1$. Since $\sigma(v_2) = v_3$, σ is color-reversing. Since p and q are odd, both reflections through perpendicular planes containing the axis of σ are color-preserving. So this construction yields SSGs with symmetry group **(xix)**. The smallest example of this type has $p = 3$ and $q = r = 1$ giving $|V_b| = 11$.

If the axes of symmetry of the parallelogram are not the diagonals, the parallelogram must be a rectangle with side SSG-coordinates of the form (p, p) or (p) leading to the next two cases.

- (b) $x_1 = 2p$, $y_2 = 2q$, $y_1 = x_2 = 0$, $x_3 = -r$ and $y_3 = p + q$, where $p > q > 0$, $r > 0$, r and $p + q$ odd. To avoid an additional reflection, we must require the $r \neq p - q$. We have $|V_b| = 2pq + r(p + q) + 1$. Since the vertices of the top and bottom rectangles have different colors, σ is color-reversing. But the reflections through perpendicular planes containing the axis of σ are color-preserving. So this construction also yields SSGs with symmetry group **(xix)**. The smallest example of this type has $p = 2$, $q = 1$ and $r = 3$ giving $|V_b| = 14$.
- (c) $x_1 = y_1 = p$, $-x_2 = y_2 = q$, $x_3 = -\frac{p+q}{2} - r$ and $y_3 = \frac{p+q}{2} - r$, where $p > q > 0$, $r > 0$ and $p + q$ even. To avoid an additional reflection, we must require the $r \neq \frac{p-q}{2}$, $r \neq \frac{p+q}{2}$ and $r \neq \frac{p+3q}{2}$.

We have $|V_b| = pq + r(p + q) + 1$. Since $\sigma(v_1) = v_3$, σ is color-reversing when $\frac{p+q}{2} - r$ is odd and color-preserving when $\frac{p+q}{2} - r$ is even. The reflections through perpendicular planes containing the axis of σ are color-reversing when p and q are odd and preserving when p and q are even. There are four possibilities:

- (1) $\frac{p+q}{2} - r$ odd, p and q odd, symmetry group **(xviii)**, smallest example $p = 7$ and $q = r = 1$ giving $|V_b| = 16$ or $p = 3$, $q = 1$ and $r = 3$ giving $|V_b| = 16$
- (2) $\frac{p+q}{2} - r$ odd, p and q even, symmetry group **(xix)**, smallest example $p = 6$, $q = 2$ and $r = 1$ giving $|V_b| = 21$.
- (3) $\frac{p+q}{2} - r$ even, p and q odd, symmetry group **(xvii)**, smallest example $p = 5$ and $q = r = 1$ giving $|V_b| = 12$.
- (4) $\frac{p+q}{2} - r$ even, p and q even, would produce an SSG with all degree-3 vertices of the same color.

\mathcal{A}_Γ is of Type **(xix)**. Since σ is color-reversing and the reflections are color-preserving, each of the three configurations is possible. The smallest possibility for (a) has $|V_b| = 11$, the smallest for (b) has 14 vertices and the smallest for (c) has 21 vertices.

\mathcal{A}_Γ is of Type **(xviii)**. Since σ and the reflections are color-reversing, only configuration (c) is possible and the smallest example has $|V_b| = 16$.

\mathcal{A}_Γ is of Type **(xvii)**. Since σ is color-preserving and the reflections are color-reversing only configuration (c) is possible and the smallest example has $|V_b| = 12$.

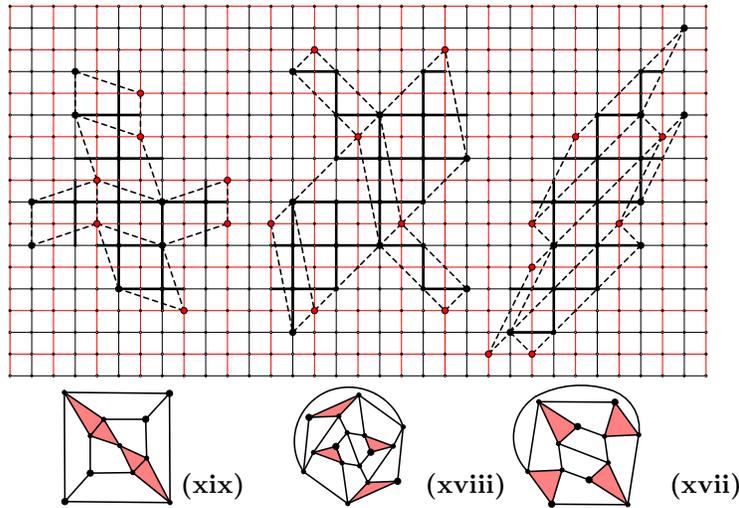


FIGURE 11. Smallest Examples (xix), (xviii) & (xvii)

The next three groups have three reflections with pairwise orthogonal equators. Let ρ_1 be one of the color-reversing reflections and let τ_1 be the composition of the other two. Then the plane of ρ_1 is perpendicular to the axis of τ_1 . Since ρ_1 is color-reversing its equator is a 45° line. Hence the rim of τ_1 must be a 45° line.

The options for the top parallelogram of τ_1 are the same three structures we discussed for the rotatory-reflection of order 4. In the first two cases (a) and (b) above, the rim of τ_1 is a horizontal line. Hence case (c) must hold: $x_1 = y_1 = p$ and $-x_2 = y_2 = q$, $p > q > 0$. The vertex (x_3, y_3) is the reflected image of the degree-3 vertex at the origin. Hence $x_3 = -r$ and $y_3 = -r$, where r is odd. To avoid additional symmetries, r is distinct from p and q . The reflections perpendicular to the edges with SSG-coordinates (p, p) and (q, q) will be color-preserving when p (respectively q) is even and color-reversing when it is odd. We have $|V_b| = pq + pr + qr + 1$.

\mathcal{A}_Γ is of Type (xvi). One color-reversing reflection, so exactly one of p, q and r is odd. The smallest example has $p = 4, q = 2$ and $r = 1$ giving $|V_b| = 15$.

\mathcal{A}_Γ is of Type (xv). Two color-reversing reflections, so exactly two of p, q and r are odd. The smallest example has $p = 3, q = 2$ and $r = 1$ giving $|V_b| = 12$.

\mathcal{A}_Γ is of Type (xiv). Three color-reversing reflections, so all of p, q and r are odd. The smallest example has $p = 5, q = 3$ and $r = 1$ giving $|V_b| = 24$.

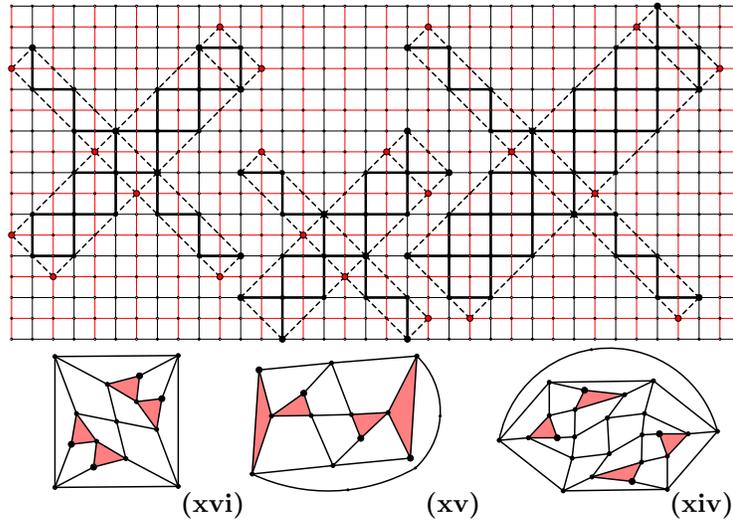


FIGURE 12. Smallest Examples (xvi), (xv) & (xiv)

The next two cases consist of a rotation σ of order 4 and four half-turns with axes perpendicular to the axis of σ and no reflections. Hence we have $x_1 = y_2 = p$ and $y_1 = -x_2 = q$ to give the square with its center as the center of the rotation of order 4. The remaining parameters x_3 and y_3 must satisfy the basic conditions, which become $\frac{p}{q} \geq \frac{y_3}{-x_3}$. Furthermore they must be different from $(-p, -q)$ to avoid a rotation of order 3. If $p = q$ or $q = 0$, x_3 and y_3 must be chosen to avoid a reflection or a rotatory-reflection of order 8 (see xxii). If p and q are odd, σ is color-reversing; if p and q are even, σ is color-preserving and x_3 and y_3 must be odd. We have $|V_b| = \frac{p^2+q^2}{2} - px_3 - qy_3 + 1$.

\mathcal{A}_Γ is of Type (xiii). One color-reversing rotation σ of order 4; so p and q are odd. The smallest example has $p = 3, q = 1, x_3 = y_3 = -1$ for $|V_b| = 10$.

\mathcal{A}_Γ is of Type (xii). One color-preserving rotation σ of order 4; so p and q are even, x_3 and y_3 odd. The smallest example has $p = 4, q = 0, x_3 = -1, y_3 = 1$ for $|V_b| = 13$.

The very last case to consider using Template #1 is:

\mathcal{A}_Γ is of Type (x). Just three half-turns with pairwise perpendicular axes. In addition to the basic conditions on the parameters they must be chosen to avoid any reflections. Since two of the half-turns must be color-reversing, we must avoid having all six parameters even or all six odd. The smallest example has $x_1 = 3, y_1 = 1, x_2 = 0, y_2 = 2, x_3 = -2$ and $y_3 = 2$ for $|V_b| = 8$.

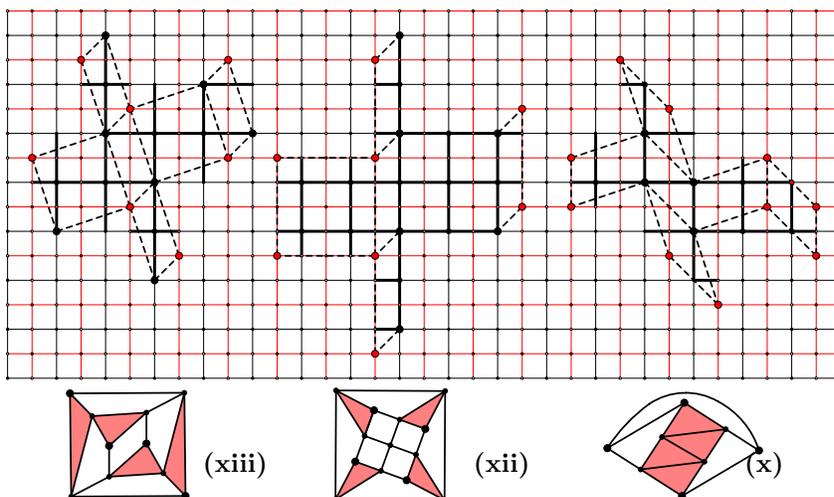
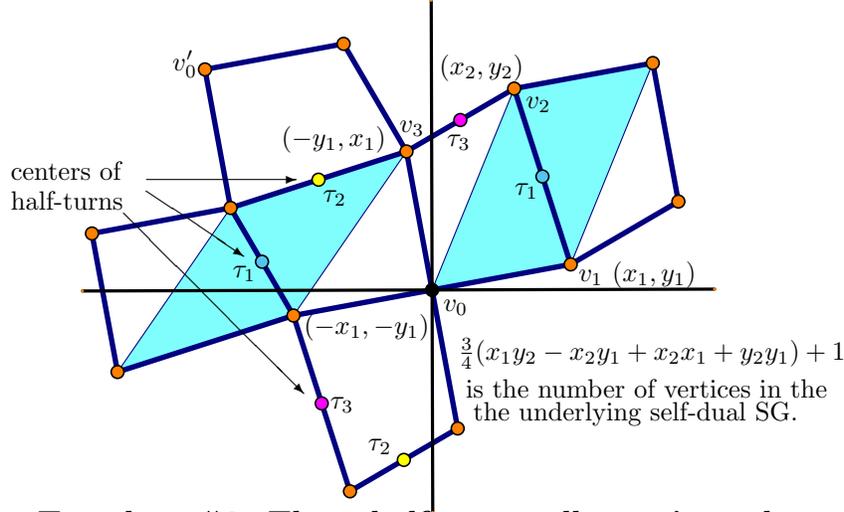


FIGURE 13. Smallest Examples (xiii), (xii) & (x)

Basic Conditions: $x_1 > 0, y_1 \geq 0$; one pair $(x_1, y_1), (x_2, y_2)$ even, the other odd; $x_1y_2 - x_2y_1 \geq 0, x_2x_1 + y_1y_2 \geq 0$ and $(x_2, y_2) \neq (x_1 - y_1, x_1 + y_1)$



Template #2: Three half-turns; all axes in a plane perpendicular to the axis of a rotation of order 3.

To avoid additional symmetries in Template #2, we must ensure that the two parallelograms corresponding to the half-turn τ_1 are not congruent. This is equivalent to ensuring that the quadrilateral v_0, v_1, v_2, v_3 is not a square and this, in turn, is equivalent to the condition $(x_2, y_2) \neq (x_1 - y_1, x_1 + y_1)$. To compute the area we note that $x_1y_2 - x_2y_1$ is the area of the parallelogram for τ_1 . Hence $\frac{1}{2}(x_1y_2 - x_2y_1)$ is the area of the triangle with vertices v_0, v_1 and v_2 . Similarly $\frac{1}{2}(x_2x_1 + y_1y_2)$ is the area of the triangle with vertices v_0, v_2 and v_3 . Their sum is then the area of the basic quadrilateral and six times their sum or $3(x_1y_2 - x_2y_1 + x_2x_1 + y_2y_1)$ is the total area.

\mathcal{A}_Γ is of Type (xxi). In order to accommodate the rotatory-reflection and the reflections through the axis of ϕ , we must have either $y_1 = 0$ and $x_2 = y_2$ or $x_1 = y_1$ and $x_2 = 0$. The smallest example is given by $x_1 = y_1 = 1, x_2 = 0$ and $y_2 = 4$ for $|V_b| = 7$.

\mathcal{A}_Γ is of Type (xx). In order to accommodate the reflection through the plane containing the axes of the half-turns and the reflections through the axis of ϕ , we must have $y_1 = 0$ and $y_2 = x_2 + x_1$. The smallest example is given by $x_1 = 2, y_1 = 0, x_2 = 1$ and $y_2 = 3$ for $|V_b| = 7$.

\mathcal{A}_Γ is of Type (xi). Avoiding additional reflections, the smallest example is given by $x_1 = 2, y_1 = 2, x_2 = 1$ and $y_2 = 3$ for $|V_b| = 10$.

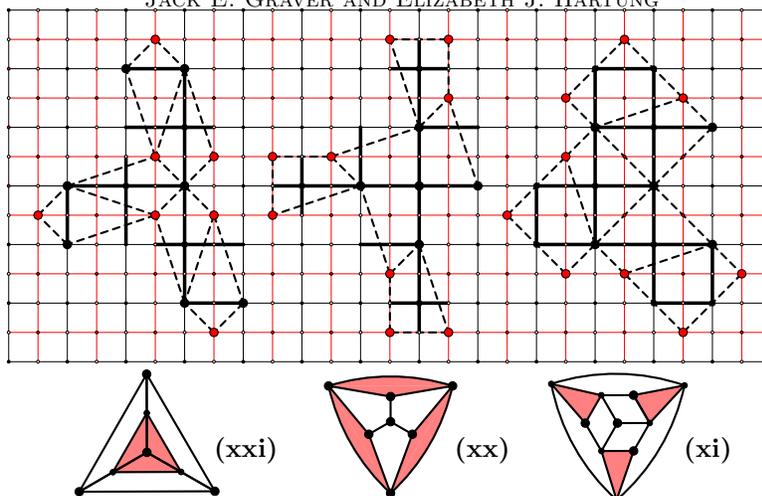
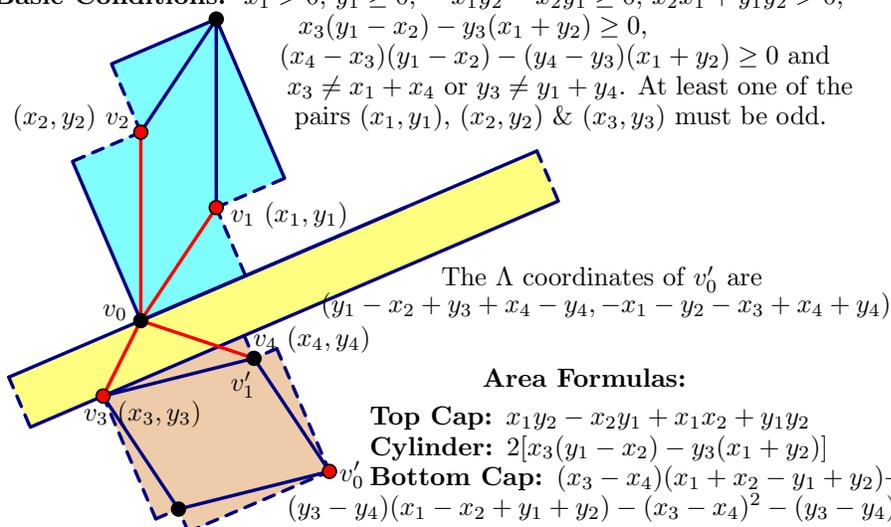


FIGURE 14. Smallest Examples (xxi), (xx) & (xi)

In constructing these smallest examples, the main problem is avoiding additional symmetries. Adding one additional symmetry will double the group size. Hence for large groups, there will be many additional symmetries and their presence will be easily detected and avoided. However, with the small groups that we encounter next, additional are easy to miss; in particular color-preserving reflections with equators containing some or all of the degree-3 vertices are very easy to miss. Our next template covers the seven groups that admit exactly one half-turn.

Basic Conditions: $x_1 > 0, y_1 \geq 0, x_1y_2 - x_2y_1 \geq 0, x_2x_1 + y_1y_2 > 0,$
 $x_3(y_1 - x_2) - y_3(x_1 + y_2) \geq 0,$
 $(x_4 - x_3)(y_1 - x_2) - (y_4 - y_3)(x_1 + y_2) \geq 0$ and
 $x_3 \neq x_1 + x_4$ or $y_3 \neq y_1 + y_4$. At least one of the
 pairs $(x_1, y_1), (x_2, y_2)$ & (x_3, y_3) must be odd.



Template #3: One half-turn.

Lemma 12. *Let Γ be an SSG that admits only one half-turn, τ . Then it may be orientated as in Template #3 and the basic conditions listed must be satisfied.*

- (a) *For groups (iv), (v), (vi), (vii) and (ix), the rims must be 45° segments with the axis of reflection the centerline of the cylinder parallel to the rims.*
- (b) *For groups (iv) and (v) the parallelograms must be non-congruent rectangles with 45° sides so that the midpoints of the v_0, v_1 segment and the v_3, v_4 segment must match up across the cylinder: $v_1 = (p, p)$, $v_2 = (-q, q)$, $v_3 = (\frac{p-s}{2} + r, \frac{p-s}{2} - r)$ and $v_4 = (\frac{p+s}{2} + r, \frac{p+s}{2} - r)$. For these groups the number of vertices in the underlying SG is $\frac{1}{2}(pq + (s + 2r)(p + q) - s^2) + 1$, where $(r, -r)$ is the width of the rim and (s, s) is the width of the bottom rectangle and $p \geq q > 0$, $s > 0$, $s \neq p, q$ and $s \geq \frac{p+q}{2}$.*
- (c) *For groups (vi), (vii), (viii) and (ix) the parallelograms must have no reflective symmetry and be the reflections of one another. For groups (vi), (vii) and (ix) the number of vertices in the underlying SG is $\frac{1}{2}(y_1^2 - x_1^2 - 2*y_1x_2) + r(y_1 - x_2) + 1$, where $(r, -r)$ is the width of the rim. An SSG with group (viii) may have a 45° equator of reflection or it may have an equator of reflection through some of the degree-3 vertices.*

Finally, we note that τ will be color-preserving when the pairs (x_1, y_1) and (x_2, y_2) have the same parity and color-reversing when they have different parity. When x_3 above is even any reflection or rotatory-reflection through the cylinder is color-preserving; when x_3 above is odd any such reflection or rotatory-reflection is color-reversing.

Proof. We may assume that the top parallelogram for τ has a vertex v_0 on the long diagonal at the origin and v_1 in the first quadrant [$x_1 > 0$, $y_1 \geq 0$]. The vertex v_2 is counterclockwise from v_1 [$x_1y_2 - x_2y_1 \geq 0$] with $\angle v_2v_0v_1$ less than 90° [$x_2x_1 + y_1y_2 > 0$]. From Lemma 8, the rim coordinates are $2(x_1 + y_2, y_1 - x_2)$. Hence the direction to vertex v_3 of the bottom parallelogram on the bottom rim must be clockwise from the top rim [$x_3(y_1 - x_2) - y_3(x_1 + y_2) \geq 0$]. Finally v_4 , vertex on the bottom parallelogram adjacent to v_3 must be on or below the bottom rim [$(x_4 - x_3)(y_1 - x_2) - (y_4 - y_3)(x_1 + y_2) \geq 0$]. Taking the vertex on the bottom parallelogram opposite v_3 as v'_0 , we see that $v'_1 = v_4$. Since the rim coordinates are $2(x_1 + y_2, y_1 - x_2)$, we compute the coordinates of v'_0 as follows:

- (1) add $\frac{1}{2}$ the rim coordinates to the coordinates of v_3 to get $(x_1 + y_2 + x_3, y_1 - x_2 + y_3)$, the coordinates of the point on the bottom rim half way around the rim;
- (2) rotate this point 90° clockwise about v_4 to get $(y_1 - x_2 + y_3 + x_4 - y_4, -x_1 - y_2 - x_3 + x_4 + y_4)$, the coordinates of v'_0 .

Since the two parallelograms for τ cannot be directly congruent, we must have $(x_4 - x_3, y_4 - y_3) \neq (x_2, y_2)$. The coordinates of the unlabeled vertex on each parallelogram are then forced. Hence we have an eight parameter family: $\{x_i, y_i, i = 1 \text{ to } 4\}$.

Groups (v) and (iv) have two reflections with planes for reflection intersecting in the axis of τ . In these cases, the parallelogram of τ must admit two perpendicular reflections. We observe that if the axes of reflections are the diagonals of the parallelogram, they must both be color-preserving. We conclude that the parallelogram must be a rectangle with the axes of reflection perpendicular to the sides. Hence all sides have SSG coordinates of the form (x) or all of the form (x, x) or $(-x, x)$. In the first case, we again get only color-preserving reflections. Hence $x_1 = y_1 = p$ and $-x_2 = y_2 = q$. The bottom parallelogram must also be a quadrilateral at a 45° angle: $v_4 - v_3 = (s, s)$. In order for the centers of the sides to line up we must have $v_3 + (\frac{s}{2}, \frac{s}{2}) = (\frac{p}{2}, \frac{p}{2}) + (r, -r)$. Solving the second equation for v_3 gives $v_3 = (\frac{p-s}{2} + r, \frac{p-s}{2} - r)$ and then solving the first for v_4 gives $v_4 = (\frac{p+s}{2} + r, \frac{p+s}{2} - r)$. The area of the top rectangle is $2pq$, the area of the cylinder is $4r(p+q)$ and the area of the bottom rectangle is $2s(p+q-s)$. The formula for the number of vertices follows at once.

The groups (vi), (vii), (viii), and (ix) all have an equator of reflection perpendicular to the axis of τ . In one case (viii), the reflection is color-preserving so the equator of reflection and the rim could be horizontal (however in this case, $y_1 - x_2 = 0$ making τ as well as the reflection color-preserving) or it could be a color-preserving equator containing several degree-3 vertices. With this one possible exception, the equator of reflection and the rim must be at a 45° angle: $x_1 + y_2 = y_1 - x_2$, giving $y_2 = y_1 - x_1 - x_2$.

In the case of the color-preserving 45° reflection v_3 has coordinates $(r, -r)$ with r even; in the case of the color-reversing reflection v_3 has coordinates $(r, -r)$ with r odd; in the case of the color-reversing rotatory-reflection v_3 has coordinates $(r + \frac{x_1+y_2}{2}, -r + \frac{x_1+y_2}{2})$ with r odd. Since the bottom parallelogram is the reflected image of the top parallelogram, $x'_1 = -x_2$ and $y'_1 = y_1$. This gives $x_4 = x_2 + x_3$ and $y_4 = y_3 - y_2$. Hence the only free parameters are x_1, y_1, x_2 and x_3 .

In terms of these parameters, the top cap, and hence the bottom cap too, has area $y_1^2 - x_1^2 - 2y_1x_2$. In both cases, the cylinder area is simply $4r(y_1 - x_2)$. Hence the number of vertices in the underlying SG is $\frac{1}{2}(y_1^2 - x_1^2 - 2 * y_1x_2) + r(y_1 - x_2) + 1$. \square

Since the SSGs with groups (iv) through (ix) have order 4, the corresponding SG has just one symmetry. When that symmetry is a half-turn or a reflection, the SG is drawn to exhibit that symmetry. However, in the case of (vii), the symmetry is the antipodal map α and we have labeled the vertices to identify this symmetry.

\mathcal{A}_Γ is of Type (ix). Since the half-turn is color-preserving x_1 and x_2 have the same parity; since the rotatory-reflection is color-reversing, $x_3 = r + \frac{x_1+y_2}{2}$ must be odd. The smallest example avoiding additional symmetries: $x_1 = 3, y_1 = 5, x_2 = -1$ and $r = 2$ for $|V_b| = 26$

\mathcal{A}_Γ is of Type (viii). τ is color-reversing; the reflection is color-preserving. The smallest example involves an equator through four degree-3 vertices: $x_1 = 2, y_1 = 6, -x_2 = y_2 = 1, x_3 = -y_3 = 1$ and $x_4 = y_4 = 3$ for $|V_b| = 13$

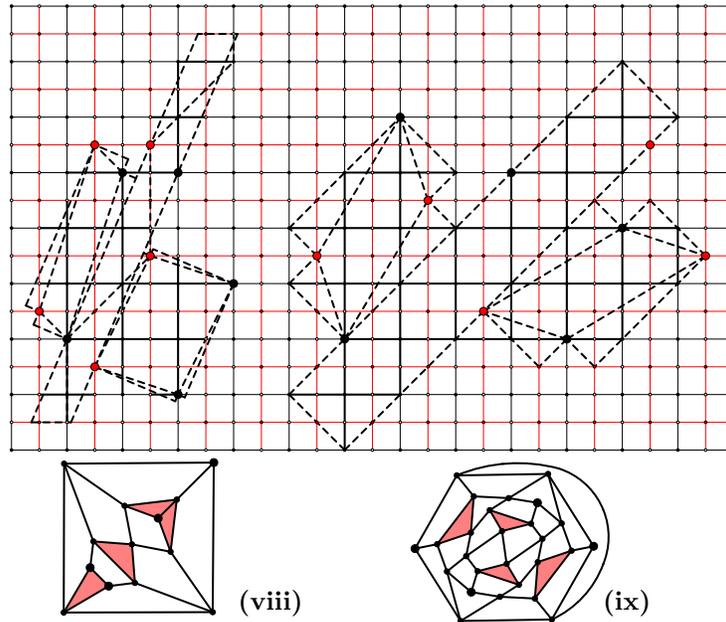


FIGURE 15. Smallest Examples (viii) & (ix)

\mathcal{A}_Γ is of Type (vii). τ is color-reversing; the reflection is color-reversing. Smallest example $x_1 = 1, y_1 = 3, x_2 = -2$ and $x_3 = 1$ for $|V_b| = 16$.

\mathcal{A}_Γ is of Type (vi). τ is color-preserving; the reflection is color-reversing. Smallest example $x_1 = 1$, $y_1 = 3$, $x_2 = -3$ and $x_3 = 1$ for $|V_b| = 20$.

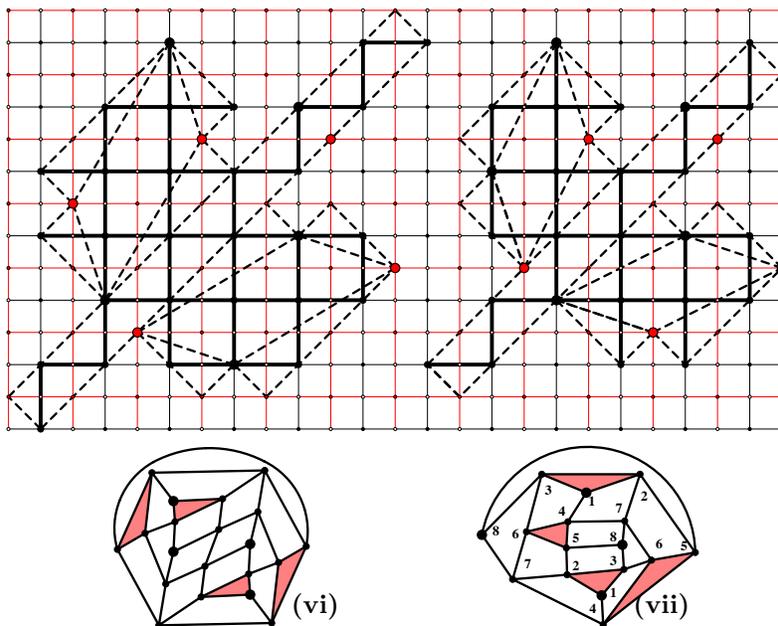


FIGURE 16. Smallest Examples (vi & (vii)

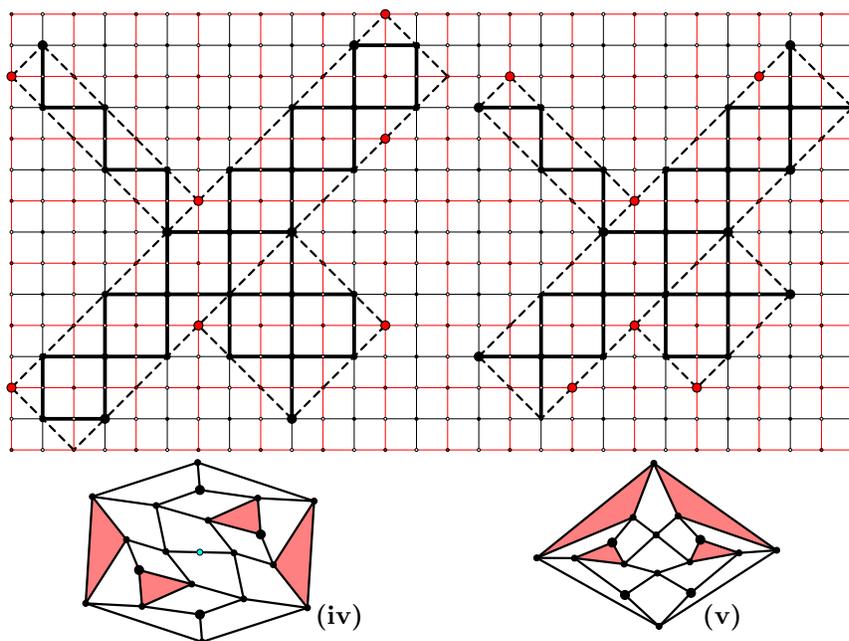


FIGURE 17. Smallest Examples (iv) & (v)

With all of the constraints in Lemma 12 (ii) the next two smallest examples are easy to construct.

\mathcal{A}_Γ is of Type (v). Smallest example $x_1 = y_1 = 1, -x_2 = y_2 = 4$. Then $x_3 = 4, y_3 = 0, x_4 = 1$ and $y_4 = -3$ giving $|V_b| = 16$.

\mathcal{A}_Γ is of Type (iv). Smallest example $x_1 = y_1 = 1, -x_2 = y_2 = 5$. Then $x_3 = 4, y_3 = 0, x_4 = 1$ and $y_4 = -3$ giving $|V_b| = 20$.

Finally, we come to the group (iii) consisting of one color-reversing half-turn and the identity. We must simply choose the parameters so that there are no additional symmetries.

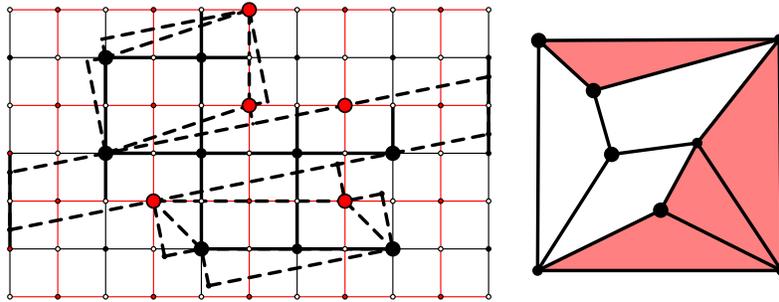
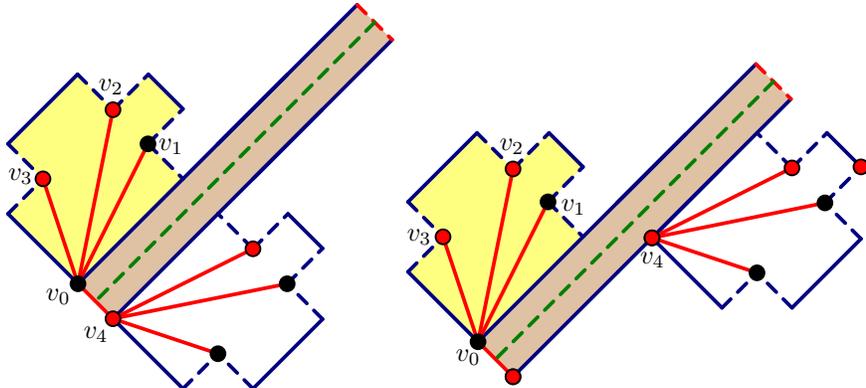


FIGURE 18. Smallest Example (iii)

\mathcal{A}_Γ is of Type (iii). Smallest example $x_1 = 3, y_1 = 1, x_2 = 0, y_2 = 2, x_3 = 1, y_3 = -1, x_4 = 0$ and $y_4 = -2$ giving $|V_b| = 8$.

Basic Conditions: $y_1 \geq x_1, x_1 y_3 - x_3 y_1 \geq 0, y_3 \geq -x_3, x_2 = y_1 - y_3, y_2 \leq y_1 + y_3, v_4 = (p, -p)$ for (i) and $(p + \frac{x_1 + y_2 - x_3}{2}, -p + \frac{x_1 + y_2 - x_3}{2})$ for (ii)



Template #4: One color-reversing opposite symmetry.

Our final template covers just the remaining two groups: (i) a single color-reversing reflection and (ii) only the color-reversing antipodal map.

As a direct result of Lemma 8, we have:

Lemma 13. *Let Γ be and SSG that admits only one symmetry: either a color-reversing reflection or the color-reversing antipodal map. Then it may be orientated as in Template #4 and the basic conditions listed must be satisfied. Furthermore in both cases*

$$|V_b| = y_1 y_3 - \frac{(y_1 - x_1)^2 + (y_3 + x_3)^2 + (y_1 + y_3 - y_2)^2}{4} + \frac{p(x_1 + y_2 - x_3)}{2}$$

\mathcal{A}_Γ is of Type (ii). Smallest example $x_1 = 2, y_1 = 4, x_2 = 1, y_2 = 5$. Then $x_3 = -1, y_3 = 3, x_4 = 7$ and $y_4 = 1$ giving $|V_b| = 22$.

\mathcal{A}_Γ is of Type (i). Smallest example $x_1 = 1, y_1 = 3, x_2 = -1, y_2 = 5$. Then $x_3 = -2, y_3 = 4, x_4 = 1$ and $y_4 = -1$ giving $|V_b| = 14$.

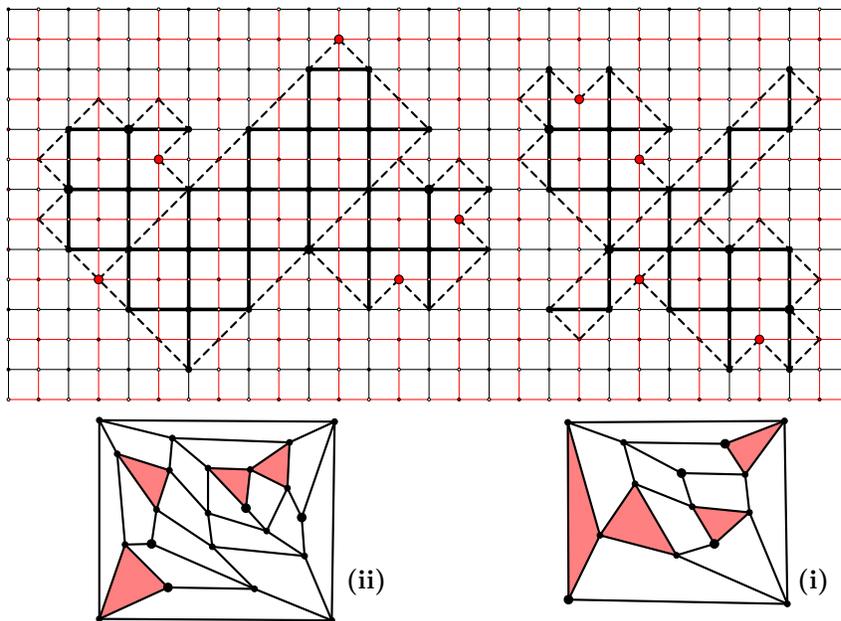


FIGURE 19. Smallest Example (ii) & (i)

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