Algebra on Review[1]

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Contents

Preface

This book concentrates on what one might call strategic algebra as opposed to tactical algebra. By tactical algebra we mean the skills and techniques needed to manipulate algebraic expressions and equations; by strategic algebra we mean the planning skills needed to employ the power of algebra to solve problems and to develop a deeper understanding of other parts of mathematics. Prior to the development of computer programs and calculators capable of symbolic manipulation, one had to be an expert algebra tactician before one could even develop their strategic algebra skills. Here, most of tactical algebra is compressed into the first three chapters and the rest of the book is devoted to strategic algebra. We are assuming that the readers have already studied tactical algebra and only need a review or are using a computer or calculator capable of symbolic manipulation or both.

Throughout this book, we stress the close relationship between algebra and the geometry via the number line or coordinate plane. We feel strongly that a geometric interpretation of an algebraic concept yields a much deeper understanding. It literally enables them to better "see what is going on."

We will also include a variety of exercises ranging from the routine to the challenging.

Jack Graver and Lawrence Lardy
Syracuse University, July 2008

Comments on This Draft.

We are in the process of revising and expanding this manuscript. We see the first four chapters as the core of the book: a review of all of high school algebra and we are concentrating on rewriting those chapters. As we make these revisions, we will periodically send updates to the instructors of the summer algebra courses.

The remaining chapters are devoted to applications of algebra. If there are any of these chapters that the instructors would like to use this summer, they should let us know and we can update those too.

CHAPTER 1

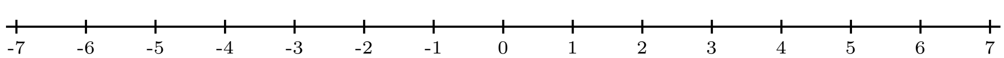
Basic Algebraic Structures

1. Numbers, Operations and the Number Line

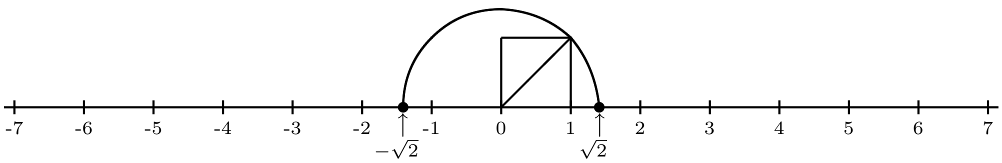
The term *number* can be interpreted in a variety of ways. So when using that term, one should be careful and the best way to do this is to employ more precise terminology. We actually work with several number systems. The simplest of these is the *natural numbers* or counting numbers, as they are sometimes called: 1, 2, 3, ... With this number system we have two basic operations: *addition* and *multiplication*. We say that this number system is *closed* under these two operations; meaning that, if you add or multiply two natural numbers, you always get another natural number. This is the very first number system known to mankind and all other number systems are built upon it. One of the early additions was to recognize 0 as a number. This enabled another step: the inclusion of the negatives of the natural numbers.

The set of natural numbers, their negatives and zero are called the *integers* or whole numbers. The integers are closed under addition, subtraction and multiplication. Some simple fractions such as $\frac{1}{2}$ and $\frac{1}{3}$ were known and used quite early. Including the full set of quotients of natural numbers $\{\frac{p}{q}\}$, where both p and q are natural numbers and q is not 0, is a number system that is closed under addition, subtraction, multiplication and, with the exception of division by 0, division. The set of all such fractions, is called the *rational numbers*. The rational numbers are closed under addition, subtraction, multiplication and division by any number except 0. Since all four of the basic mathematical operations are available for the rational numbers, the arithmetic of the rational numbers is very powerful.

While the rational number system is an ideal system in which to do algebra and quite adequate for many applications, bookkeeping for example, it is not sufficient to do the algebraic computations that arise out of geometry. It is not possible to represent all distances by rational numbers! The Greeks were well aware of this. They proved very early that the diagonal of the square with side length 1 is $\sqrt{2}$, the square root of 2, and that $\sqrt{2}$ is not a rational number. So the number system we really want to work in is the set of all possible lengths and their negatives. This system is best represented by the number line.



The number on the number line that represents a length is the end point of the segment of that length stretching from 0 to the right; the end point of the segment of that length stretching from 0 to the left is its negative. We illustrate with the square root of 2.



This number system is called the *real numbers* and this way of picturing them the *number line*. The real numbers contain all of the rational numbers. The real numbers that are not rational numbers are called *irrational numbers*

Representing a real number can be somewhat of a problem. Of course, if the real number is also an integer or rational number, we have the usual representation; for example -17 or $\frac{86}{5}$. However, most real numbers are not rational numbers. Some of them have special names like $\sqrt{2}$ or π; but, most do not. The simplest method of representing a real number is by its *decimal expansion*. For example, -17 is -17.000..., the ending string of 0s goes on forever; $\frac{86}{5} = 17.200\ldots$; $\sqrt{2} = 1.41421\ldots$, here the dots indicate that the string of digits goes on forever. Integers have decimal expansions with all digits to the right of the decimal point equal to zero. Rational numbers have decimal expansions with the digits to the right of the decimal point ending in a repeating pattern; for example; $\frac{229}{260} = 0.88076923076923076923\cdots = 0.88\overline{076923}$, the overline indicates the segment of digits to be repeated. The irrational real numbers have decimal expansions that "go on forever" without a repeating pattern. It is quite clear from this that there are an infinite number of irrational numbers.

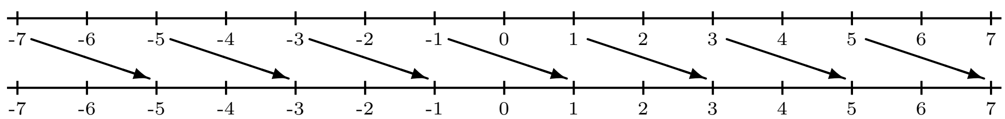
In reporting numbers one has several choices: $\frac{10}{4}$, $\frac{5}{2}$, $2\frac{1}{2}$ and 2.5 all represent the same number. $\frac{5}{2}$ and 2.5 are the most commonly used; $\frac{10}{4}$ is useful for making some computations: $\frac{5}{2} + \frac{1}{4} = \frac{10}{4} + \frac{1}{4} = \frac{11}{4}$. We will seldom use the mixed fraction form $2\frac{1}{2}$ since it may be misinterpreted as $2 \times \frac{1}{2} = 1$. For irrational numbers like π, we will write 3.14 or 3.14159 or 3.14159265359 with the understanding that these are only approximations of π. A natural question to ask is "how can we tell if a real number is rational or irrational?" If we compute any number of its digits, how could we know that it would not start a repeating pattern if we had computed the next few? In general, it may be very hard to decide if a number is rational or irrational. It was first proved that π was irrational in the late 1700s. However, it is rather easy to see that $\sqrt{2}$ is not rational (and therefore irrational).

Exercise 1.1. In this exercise, we outline a proof that $\sqrt{2}$ is irrational; you are to fill in the details. It is a proof by contradiction, that is, we will assume that $\sqrt{2}$ is rational and show that that assumption leads us to an impossible equality. The basic fact that we will use is a fundamental property of the natural numbers: each natural number has a unique factorization into primes. For example $360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5$ can be written no other way as a product of primes (except to change the order in which they are multiplied). We start by assuming that $\sqrt{2} = \frac{p}{q}$, where p and q are natural numbers.

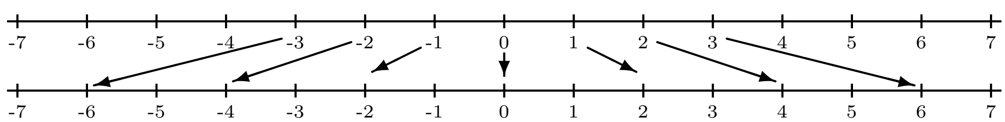
(i) Then $p = \sqrt{2}q$.
(ii) Hence $p^2 = 2q^2$.
(iii) p^2 has an even number (perhaps 0) of 2s in its prime factorization.
(iv) $2q^2$ has an odd number (perhaps 1) of 2s in its prime factorization.
(v) Thus, $p^2 = 2q^2$ is impossible.

Like the rational numbers, the real numbers are closed under addition, subtraction, multiplication and division by any number except 0. The Greeks showed this by giving simple methods for constructing a length equal to the sum or difference of two lengths, for a length equal to the product of two lengths and for a length equal to the quotient of two non-zero lengths. These are described for the interested reader in an exercise at the end of this section. Since the integers belong to the real numbers and the real numbers are closed under division, all rational numbers belong to the set of real numbers.

The number line gives us a nice geometric way of viewing our basic operations of adding, subtracting, multiplying and dividing. The result of adding 2 to 3 is 5, the number 2 units to the right of 3. In general adding 2 to any number results in the number 2 units to its right. So we can think of adding 2 as shifting the number line 2 units to the right. Adding -2 or subtracting 2 shifts the number line 2 units to the left.



Multiplying by 2 doubles the distance of each number from 0:



So we may think of multiplying by 2 as stretching or magnifying the number line by a factor of 2 about 0. Dividing by 2 is the same as multiplying by $\frac{1}{2}$ and may be thought as shrinking or contracting the number line by a factor of $\frac{1}{2}$ about 0. Finally, multiplying each number by -1, i.e. negating each number, simply flips the number line over 0. To indicate that we are thinking of these operations as transformations of the line, we can write $x \to (x+2)$, where x represents any real number. Using this notation the other transformations we have just introduced are $x \to (x-2)$, $x \to 2x$, $x \to \frac{1}{2}x$ and $x \to -x$. We use this notation in the next exercise

Exercise 1.2. Describe each of the following transformations algebraically and geometrically. In each case check what happens to the number 0.

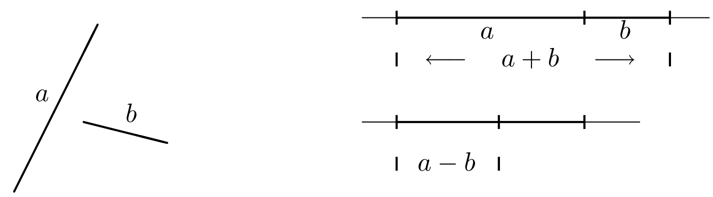
(i) $x \to (x + \frac{1}{2})$;
(ii) $x \to \frac{4}{7}x$;
(iii) $x \to -\frac{4}{7}x$;
(iv) $x \to (3x + 2)$;
(v) $x \to (-3x + 2)$.

Exercise 1.3. Give the transformation notation for each of the following operations or sequence of operations. In each case check what happens to the number 0.

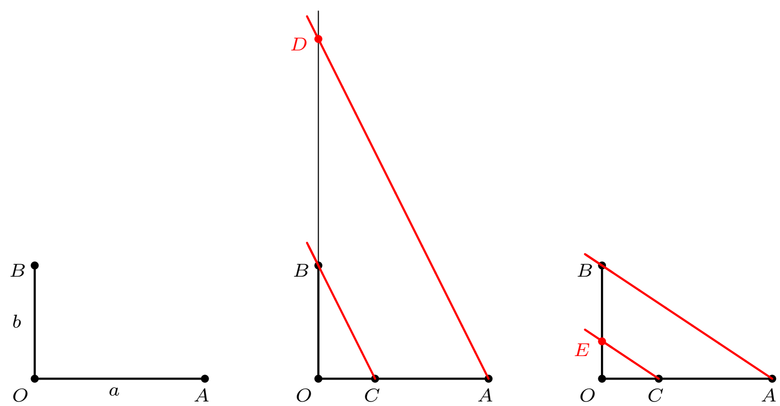
(i) multiply each number by 7;
(ii) multiply each number by 7 and then add 2;
(iii) add 2 to each number and then multiply each number by 7;

(iv) divide each number by 7;
(v) divide each number by 7 and then subtract 2;
(vi) subtract 2 from each number and then divide each number by 7;

Since lengths may be copied from one segment to another using a pair of compasses, addition and subtraction are easy to carry out geometrically. we Illustrate this procedure by constructing $a+b$ and $a-b$ where a and b are the lengths pictured below:



The geometric construction of the product of two lengths a and b goes like this: First reposition the two lengths so that they have a common end point and are perpendicular to one another as indicated on the left in the figure below. Next mark off one unit from O along segment a and denote the other endpoint of that segment by C.



To multiply b by a construct the line through C and B and then the parallel line through A. Let D denote the intersection of this line with the line through O and B. In the next exercise you will be asked to prove that the length of OD is b times a. To divide b by a construct the line through A and B and then the parallel line through C. Let E denote the intersection of this line with the line through O and B. In the next exercise you will be asked to prove that the length of OE is b divided by a.

Exercise 1.4.
(i) Prove that the length of OD is b times a; remember that OC has length 1.
(ii) Prove that the length of OE is b divided by a.

Exercise 1.5. In verifying these constructions, we assumed that $a > 1$. Make and verify the constructions in the case that $a < 1$.

A Note About Notation. The symbols $+$ and $-$ are standard for addition and subtraction. Multiplication, on the other hand, is represented in several different ways. For example, the product of numbers x and y can be written as: $x \times y$, $x \cdot y$, $x * y$ or simply xy. Similarly, the quotient, x divided by y, can be written $x \div y$, x/y or $\frac{x}{y}$. One should take care not to confuse the multiplication symbol $\times$ and the letter x, a variable standing for some unknown number. There is a special notation used when a number is multiplied times itself: $x \times x = x^2$, $x \times x \times x = x^3$, and so on.

A Note About Parentheses. Each of these operations is defined as a way of combining exactly two numbers. So when confronted with an expression like $3 \times 4 + 5$, we have to be told which operation to carry out first. To give these instructions we use parentheses; $(3\times4)+5$ or $3\times(4+5)$. All operations within parentheses should be carried out before the result is combined with numbers outside the parentheses. Hence in the first case, we have $(3 \times 4) + 5 = 12 + 5 = 17$ and, in the second case, $3 \times (4 + 5) = 3 \times 9 = 27$. In some expressions, the way the parentheses are placed makes no difference. For example, $3 + 7 + 6 = 16$ no matter "how you slice it." Also in some case, there are conventions when parentheses are absent. Generally, multiplication takes precedence over addition; so $3 \times 4 + 5$ is understood to mean $(3 \times 4) + 5$ and not $3 \times (4 + 5)$. In complicated expressions such as $(2(3 + 4) - 5)(3 - 2(4 - 1))$, we start from the inside and work out:

$$(2(3+4)-5)(3-2(4-1)) = (2\times7-5)(3-2\times3) = (14-5)(3-6) = 9\times(-3) = -27.$$

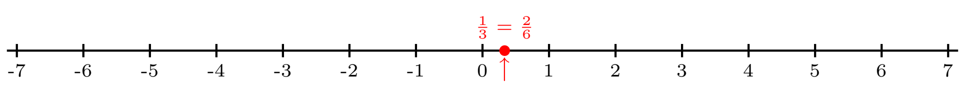
Exercise 1.6. Compute each of the following.

(i) $(15 - 5(8 - 6))/5 + 4(7/2 + 3 \times (-2))$
(ii) $(4 - 3 + 2(4 - 3))/(5(4 - 3) - 3(4 - 5))$
(iii) $(4 - 6)/(3 + 2) + (3 + 2)/(4 - 6)$

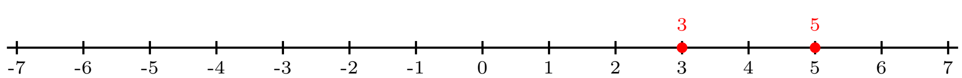
2. Simple Algebraic Sentences

The very simplest of algebraic sentences should really be called numeric sentences:

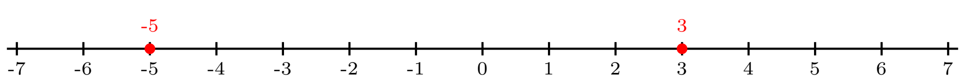
- $\frac{1}{3} = \frac{2}{6}$, the numbers $\frac{1}{3}$ and $\frac{2}{6}$ are *equal*, that is, have the same value or occupy the same position on the number line.



- $3 < 5$, the number 3 is *less than* the number 5. Geometrically, this means that 3 appears to the left of 5 on the number line.



- $3 > -5$, the number 3 is *greater than* the number -5. Geometrically, this means that 3 appears to the right of -5 on the number line.



Like sentences in English, numerical and algebraic sentences may be false. For example, the sentences $3 = 5$ and $3 > 5$ are both false. The symbol $\neq$ is often used to indicate that the corresponding equality is false. For example, the algebraic sentence $3 \neq 5$, which is read as 3 *is not equal to* 5, is a true statement. This is a simple way of saying that the sentence $3 = 5$ is false.

Numeric sentences can be complicated by the use of arithmetic operations. The operations of addition and multiplication are denoted in the usual way; $+$ for addition and $\times$ for multiplication. As we pointed out above, when both operations occur in a statement, it is understood that multiplication takes precedence over addition *unless the parentheses indicate otherwise.* That is, multiplication is performed first.

EXERCISE 1.7. In each of the following cases decide if each of the listed sentences is true or false.

(i) $5,996,427 + 5 = 5,996,427 + 6$,
$5,996,427 + 5 < 5,996,427 + 6$,
$5,996,427 + 5 > 5,996,427 + 6$.
(ii) $299 \times 31 - 7 < 299 \times 31 - 5$,
$299 \times 31 - 7 = 299 \times 31 - 5$,
$299 \times 31 - 7 > 299 \times 31 - 5$.
(iii) $3 \times 5 - 11 < 3 \times 6 - 11$,
$3 \times 5 - 11 = 3 \times 6 - 11$,
$3 \times 5 - 11 > 3 \times 6 - 11$,
$3 \times 5 - 11 \neq 3 \times 6 - 11$.
(iv) $7 \times 3 + 7 \times 2 = 7 \times 5$
$7 \times (3 + 7) \times 2 > 100$
$7 \times (3 + 7) \times 2 < 100$
$7 \times 3 - 5 = 5 - 7 \times 3$

EXERCISE 1.8. In each of the following cases decide if each of the listed sentences is true or false.

(i) $3 + 5 = 8$
(ii) $3 + 8 = 17 - 5$
(iii) $3 \times 5 - 3 < 10$
$3 \times 5 - 3 = 10$
$3 \times 5 - 3 > 10$
(iv) $3 \times 7 + 12 = 12 + 7 \times 3$
$3 \times 7 + 2 > 1 + 7 \times 3$
(v) $2 \times (4 + 5) = 2 \times 4 + 2 \times 5$
$2 \times (4 + 5) = 2 \times 4 + 5$

EXERCISE 1.9. In each of the following cases fill in the blank with a single number that will make the sentence true.

(i) $\_\_\_\_ + 5 = 17$
(ii) $29 - \_\_\_\_ = 17 - 5$
(iii) $3 \times 5 - \_\_\_\_ < 10$
(iv) $7 \times 3 + \_\_\_\_ = 28$
(v) $\frac{3}{2} \times 6 - \_\_\_\_ < 5$
(vi) $2 + 2 \times 3 + \_\_\_\_ = 9$

3. Variables

In Exercise 1.9, we used a blank to indicate a number to be computed. It is more convenient to use a letter to represent such an unknown quantity. Such a letter is called a *variable,* because it can represent various values. Rewriting that exercise with variables, we get (Of course, other choices for letters could be used.):

(i) $x + 5 = 17$
(ii) $29 - w = 17 - 5$
(iii) $3 \times 5 - a < 10$
(iv) $7 \times 3 + y = 28$
(v) $\frac{3}{2} \times 6 - v < 5$
(vi) $2 + 2 \times 3 + s = 9$

We will use the term *algebraic sentences* to include number sentences and those sentences involving numbers and variables.

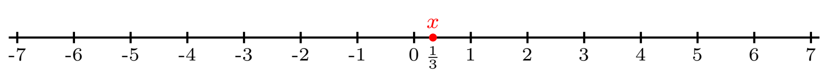
Perhaps the simplest of all *algebraic sentences* are those of the form "$x = 7$" or "$a = \frac{1}{5}$." In fact, one of the basic goals in algebra is to reduce a complicated algebraic sentence to one such simple sentence. For example, given the sentence "$2x - 3 = x + 4$", we can carry out a sequence of algebraic steps to show that it is equivalent to the simpler sentence "$x = 7$." By that we mean the two sentences "say exactly the same thing" much as the English sentences "Having red hair is one of the traits that helps distinguish Fred from his classmates." and "Fred's hair is red" say basically the same thing.

Exercise 1.10. In each of the following cases compute a value for the variable that makes the sentence true. Give your answer in a simple algebraic sentence.

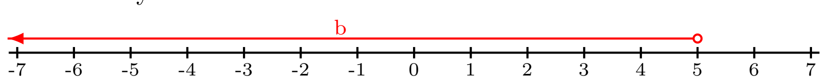
(i) $x + 5 = 17$
(ii) $29 - w = 17 - 5$
(iii) $7 \times 3 + y = 28$
(iv) $2 + 2 \times 3 + s = 9$

We call the set of values for the variables that make an algebraic sentence true the *solution set* for that algebraic sentence. Just like the simplest numerical sentences, the solution sets for the simplest algebraic sentences may be visualized on the number line.

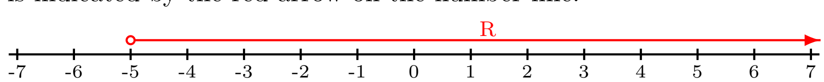
- $x = \frac{1}{3}$, the variable x equals the number $\frac{1}{3}$, that is, the only value for the variable x that makes this a true algebraic sentence is $\frac{1}{3}$, or occupy the same position on the number line.



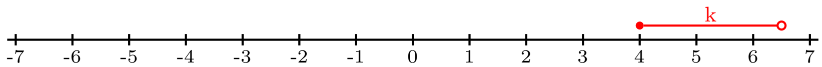
- $b < 5$, the values for b that make this a true algebraic sentence are all numbers that are to the left of 5 on the number line. This solution set is indicated by the red arrow on the number line.



- $R > -5$, the values for R that make this a true algebraic sentence are all numbers that are to the right of -5 on the number line. This solution set is indicated by the red arrow on the number line.

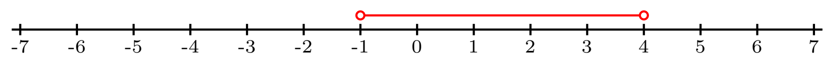


It is often the case that one wants to find the values of a variable that make two or more algebraic sentences true. For example, we might want to identify the values of the variable k that make both of the sentences $k \geq 4$ and $k < 6\frac{1}{2}$ true. (Note: The symbol $\geq$ is used to indicate that the quantity to the left of the symbol is greater than *or* equal to the quantity to the right.) These are the numbers that lie to the left of $6\frac{1}{2}$ and are either equal to 4 or lie to the right of 4. To indicate that 4 is included in the solution set, the circle representing 4 is filled in:

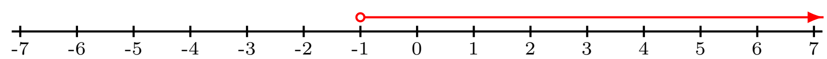


EXERCISE 1.11. In each case, write an algebraic sentence or sentences that have the set of numbers pictured in red as the solution set. Also write these sentences in words.

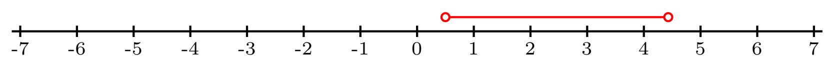
(i)



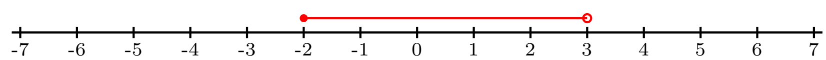
(ii)



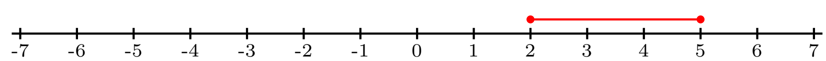
(iii)



(iv)

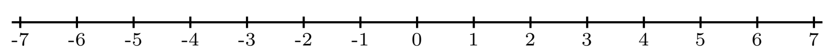


(v)

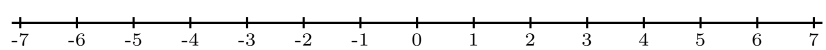


EXERCISE 1.12. In each case indicate on the number line the set of values for the variable that make the algebraic sentence(s) valid.

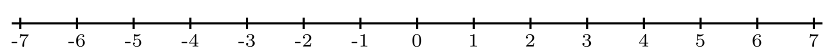
(i) $x > -\frac{1}{2}$ and $x < 3$



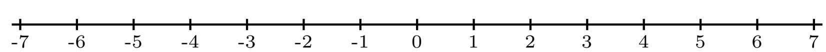
(ii) $x < -\frac{1}{2}$ or $x > 3$



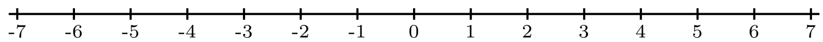
(iii) $x < -\frac{1}{2}$ and $x < 3$



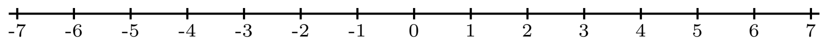
(iv) $x < -\frac{1}{2}$ or $x < 3$



(v) $x > -1$ and $x \leq 3$

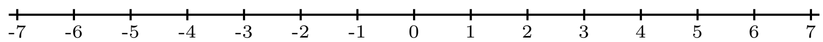


(vi) $x > -2$ and $x \neq 0$ and $x \neq 1$

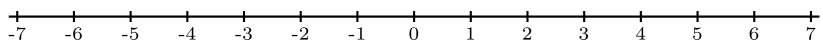


Exercise 1.13. Algebraic sentences can be complicated by the use of algebraic operations. In each case indicate on the number line the set of values for the variable that make the algebraic sentence(s) valid.

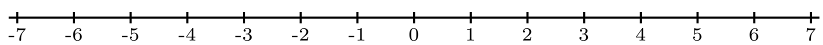
(i) $3x + 5 = 8$



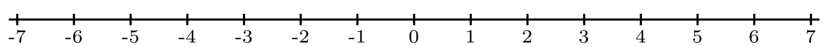
(ii) $3 + x = 14 - 5$



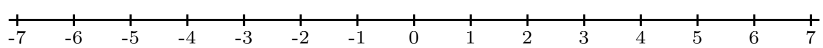
(iii) $3x - 2 < 10$



(iv) $2 - 3x < 8$

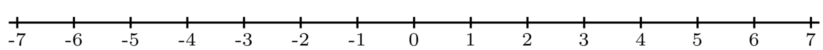


(v) $2x + 5 \leq 7$ and $3 - x \leq 7$

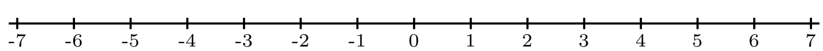


Exercise 1.14. In each case indicate on the number line the set of values for the variable that make the algebraic sentence(s) valid. Then write a simpler algebraic sentence that has the same solution set.

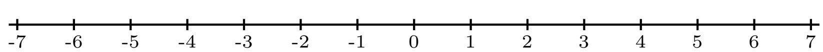
(i) $5,996,427 + x = 5,996,427 + 6,$



(ii) $299 \times 31 - g < 299 \times 31 + 5,$

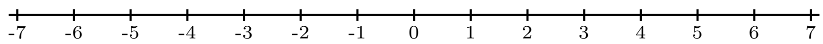


(iii) $3 \times w - 4 \times 11 < 3 \times 6 - 4 \times 11,$



(iv) $\frac{1}{2} \times p - 5 \times 12 < \frac{1}{2} \times 7 - 5 \times 12,$

(v) $2 \times s + 8 \times 4 \geq 6 + 8 \times 4,$



4. Modifying Algebraic Sentences

In Exercises 1.7 and 1.14, you saw some methods for simplifying numerical and algebraic sentences. In this section we will formalize these and list a few more. Each of the operations in the following list will not change the validity of a numerical equation and they will not change the solution set of an algebraic equation.

- add or subtract the same number to each side of an **equation or inequality**;
- multiply or divide each side of of an **equation** by the same non-zero number;
- multiply or divide both each side of an **inequality** by the same positive number;
- multiply or divide each side of of an **inequality** by the same negative number and reverse the direction of the inequality.

Exercise 1.15. Consider the following questions.

(i) What would happen if you were to multiply both sides of an **equality** by 0?
(ii) What would happen if you were to multiply both sides of an **inequality** by 0?

To complete the list of exclude cases, let's consider division by 0. What is the result when a number, say 2 to be specific, is divided by 0? To answer this question, recall that the operation of division of a number a by a number b should yield a number c such that $a = b \times c$. Thus, $2 \div 0$ should be a number c such that $2 = 0 \times c$. But c is supposed to be a number, so $0 \times c = 0$ and $2 \neq 0$. Therefore we conclude that division by 0 is not meaningful and say that it is an undefined operation. Another, very important and very useful property of 0 is described below in two different ways:

- The product of any two non-zero numbers is non-zero.
- If $a \times b = 0$, then either $a = 0$ or $b = 0$.

Each side of an algebraic equation or inequality is called an *algebraic expression.* We can expand our list of operations that will not change the validity of a numerical equation and they will not change the solution set of an algebraic equation.

- add or subtract the same algebraic expression to both sides of an equation or inequality;
- multiply or divide both sides of an equation by an algebraic expression that is never 0;
- multiply or divide both sides of an inequality by an algebraic expression that is always positive;
- multiply or divide both sides of an inequality by an algebraic expression that is always negative and reverse the direction of the inequality.

Since many algebraic expressions can be positive for some values of the variable, negative for other values and zero for other values multiplying or dividing both sides of an inequality by such an algebraic expression can change the solution set. Such cases must be analyzed carefully.

Using the properties listed above, we can often achieve the goal mentioned earlier: reduce complicated algebraic equations to simpler algebraic equations. For example, consider $7x - 7 = 5x + 3$.

(i) Adding 7 to each side, $7x - 7 + 7 = 5x + 3 + 7$,
gives the equivalent equation $7x = 5x + 10$.
(ii) Subtracting $5x$ from each side, $7x - 5x = 5x + 10 - 5x$
gives the equivalent equation $2x = 10$.
(iii) Finally, dividing both sides by 2, $\dfrac{2x}{2} = \dfrac{10}{2}$
gives the simpler equivalent equation $x = 5$.

It is always a good idea to check that the solution(s) to the simple equation are actually solutions to the original equation. If we replace x by 5 in $7x - 7 = 5x + 3$, we get $7 \times 5 - 7 = 5 \times 5 + 3$ or $35 - 7 = 25 + 3$ or $28 = 28$. Check!

EXERCISE 1.16. Simplify each of the following algebraic sentences.

(i) $3 \times W + 7 = 17 - W$
(ii) $3 \times W + 7 < 17 - W$
(iii) $4 \times x - 23 \geq 1 - 2 \times x$
(iv) $2 \times p + 5 < 7 - 2 \times p$
(v) $4x - 5 + x^2 - (x + 3)^2 + x + 49$
(vi) $4x - 5 + x^2 - (x - 3)^2 + 2x - 5$
(vii) $4x - 5 + x^2 - (x - 3)^2 + 2x + 5$

Now consider the equation $(p - 7) \times (p + 3) = 0$ This algebraic sentence can be simplified using the fact that when the product of two numbers is zero then at least one of the numbers must be zero. Since algebraic expressions stand for numbers, this is true for algebraic expressions too. Hence, $(p - 7) \times (p + 3) = 0$ may be rewritten in terms of two simpler algebraic sentences as:

Either $(p - 7) = 0$ *or* $(p + 3) = 0$.

Each of these two algebraic sentences can be further simplified and we have $(p - 7) \times (p + 3) = 0$ is equivalent to: *Either* $p = 7$ *or* $p = -3$.

EXERCISE 1.17. Simplify each of the following algebraic sentences.

(i) $(3 - W) \times W = -3W$
(ii) $(3 - W) \times (W + 7) = 0$
(iii) $4 \times (3x - 21) = x \times (3x - 21)$
(iv) $4 \times (3x - 21) = x \times (x - 7)$
(v) $(x + 3)^2 = 0$
(vi) $(5 + x)^2 \times (x - 3)^2 = 0$

Historical Note. Around 1650 BC, a man named Ahmes recorded the following problem in the Rhind Papyrus (now called the Ahmes Papyrus): "A quantity added to a quarter of itself makes 15". Translated to an equation we have $x + \frac{x}{4} = 15$. Ahmes used trial and error to obtain a solution. How might he do that?

We can push this line of reasoning a bit further by noting that the product of two positive numbers is a positive number, the product of two negative numbers is a positive number and the product of one positive number and a negative number is a negative number. Suppose $(x - 3)(x + 4) > 0$, then either $(x - 3) > 0$ and $(x + 4) > 0$ or $(x - 3) < 0$ and $(x + 4) < 0$. Simplifying both $(x - 3) > 0$ and $(x + 4) > 0$, we see that in this case $x > 3$ and, simplifying both $(x - 3) < 0$ and

$(x + 4) < 0$, we have $x < -4$. Hence, if $(x - 3)(x + 4) > 0$, then either $x > 3$ or $x < -4$. Now suppose $(x - 3)(x + 4) < 0$, then either $(x - 3) > 0$ and $(x + 4) < 0$ or $(x - 3) < 0$ and $(x + 4) < 0$. Simplifying both $(x - 3) > 0$ and $(x + 4) < 0$, we see that, in this case, $x > 3$ and $x < -4$, which is impossible. Simplifying both $(x - 3) < 0$ and $(x + 4) > 0$, we have $-4 < x < 3$. Hence, if $(x - 3)(x + 4) < 0$, then $-4 < x < 3$.

EXERCISE 1.18. Simplify each of the following algebraic sentences.

(i) $(3 - W) \times W < -3W$
(ii) $(3 - W) \times (W + 7) > 0$
(iii) $4 \times (3x - 21) < x \times (3x - 21)$
(iv) $4 \times (3x - 21) > x \times (x - 7)$
(v) $(x + 3)^2 > 0$
(vi) $(x + 3)^2 < 0$
(vii) $(5 + x)^2 \times (x - 3)^2 < 0$

We close this section by including an important additional method for modifying algebraic expressions and equations: the method of *substitution.* The idea is simple, we replace a variable by an algebraic expression through out the original expression or equation. For example, Suppose that we are given $\frac{3}{4}x - \frac{9}{4} = \frac{2}{3}x - 1$ to solve. One possible approach is to replace the variable x by the expression $(12z+3)$:

$$\frac{3}{4}(12z + 3) - \frac{9}{4} = \frac{2}{3}(12z + 3) - 1$$

Simplifying, we have $9z = 8z + 1$ or $z = 1$. Setting $x = 12z + 3$ then gives $x = 15$. We easily check that $x = 15$ works in the original equation $\frac{3}{4}x - \frac{9}{4} = \frac{2}{3}x - 1$.

EXERCISE 1.19. In each case, make the indicated substitution and simplify the resulting algebraic sentence.

(i) Let $x = (3y - 6)$ in $\frac{x}{3} - 5 = \frac{5}{3}x - 2$
(ii) Let $x = 30y$ in $\frac{x}{3} - \frac{x}{5} = \frac{1}{4}x^2$

5. Multiple Variables

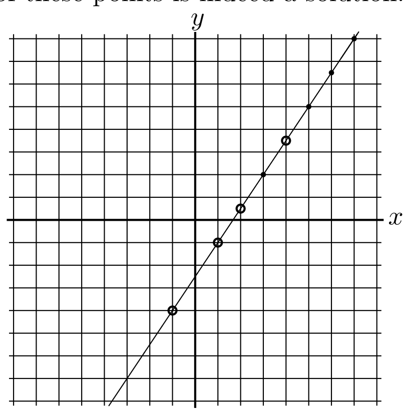
Sometimes it is useful to consider algebraic sentences that involve two or more variables. For example, $3x - 2 = 3 + 2y$. By a *solution* to this equation we mean two numbers, a value for x and a value for y, which when substituted for x and y respectively, the result is a true numerical sentence. Letting $x = 3$ and $y = 2$ in the example gives the sentence $9 - 2 = 3 + 4$ and therefore this pair of values is a solution. In general, an algebraic sentence with more that one variable can have many solutions. For the example, $x = 5$ and $y = 5$ is another solution, for $3x - 2 = 3 + 2y$ as is $x = 6$ and $y = 6\frac{1}{2}$. There is an easy way to describe all solutions to $3x - 2 = 3 + 2y$. We "solve the equation" for x or for y:

(i) Adding 2 to each side gives $3x = 5 + 2y$;
(ii) then, dividing both sides by 3 gives, $x = \frac{5+2y}{3}$.

Now select *any* number for y, say 8, replace y by 8 and simplify to get $x = \frac{5+2\times8}{3} = \frac{21}{3} = 7$. So $x = 7$, $y = 8$ is a solution. We get all possible solutions by taking all possible values for y. Of course, we could have solved for y "in terms of x:"

(i) Subtracting 3 from each side gives $3x - 5 = 2y$;
(ii) then, dividing both sides by 2 gives, $\frac{3x-5}{2} = y$.

Here it is customary to rewrite the equation as $y = \frac{3x-5}{2}$. Taking $x = 7$ gives $y = 8$, the solution we had above. We write our solutions as a pair $(7, 8)$ where the order is always the x value followed by the y value. Taking x equal to 3, 5 and 6, give us three more solutions $(3, 2)$, $(5, 5)$ and $(6, \frac{13}{2})$. We picture or *plot* these four solutions as points in the coordinate plane. And we note that they all lie on a straight line. Furthermore, the converse is also true: every point on the line corresponds to a solution for $3x - 2 = 3 + 2y$. We have circled four other points on this line: $(x = -1, y = -4)$, $(x = 1, y = -1)$, $(x = 2, y = \frac{1}{2})$, $(x = 4, y = \frac{7}{2})$. You should check that each of these points is indeed a solution.

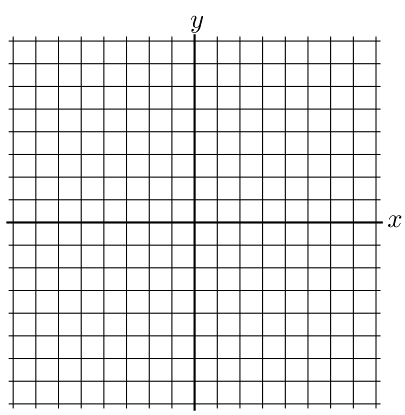


Exercise 1.20. Solve each of the following equations for one of the variables. Then list several solutions.

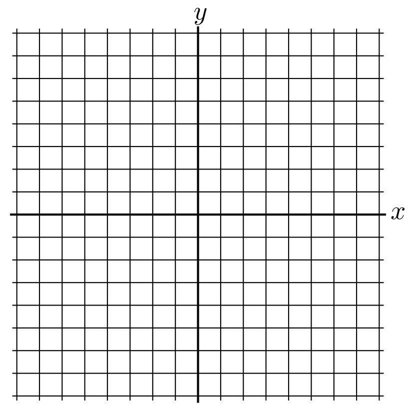
(i) $x + 2y + 1 = 7$
(ii) $x + 2y + 4 = 7x$
(iii) $2s + 5t = s + 5$
(iv) $3u - 5v = 2u + v + 1$

Exercise 1.21. In each of the following cases, solve for one variable in terms of the other and graph the solution set.

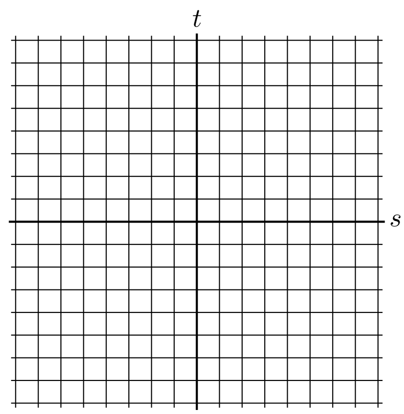
(i) $x + 2y + 1 = 7$



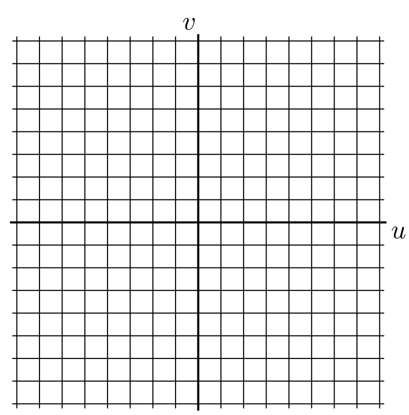
(ii) $x + 2y + 4 = 7x$



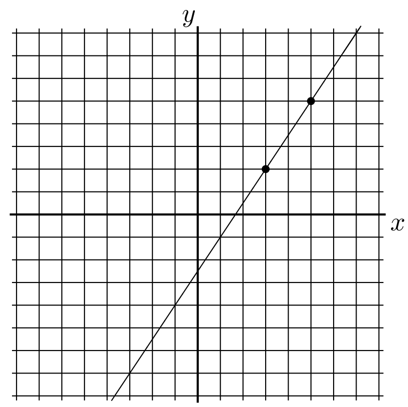
(iii) $2s + 5t = t + 4$



(iv) $u - 4v + 6 = 9 - 2u - v$

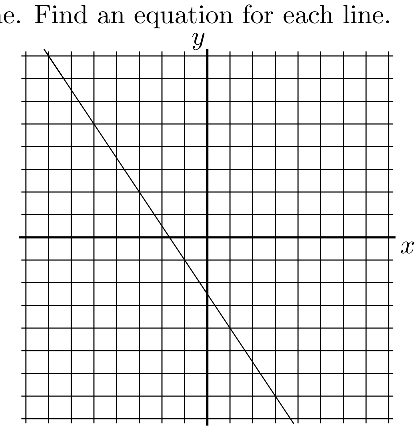


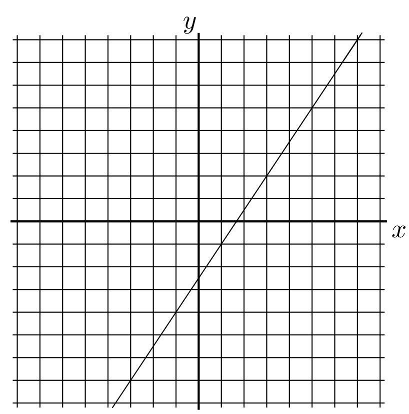
Suppose we have the graph of a line in the x, y-plane. Can we find an equation for which this line is the solution set? In general there are many different equations for the same line; but each such equation can be put in the form $ax + by = c$. Our task is to use the information we have about the line to determine values for a, b and c such that the equation $ax + by = c$ has the line as its solution set. To be specific, consider the following graph:



Each point on the graph will give us an equation involving a, b and c. For example, the point $(x = 3, y = 2)$ on the line gives the equation $a \times 3 + b \times 2 = c$ and the point $(x = 5, y = 5)$ yields $a \times 5 + b \times 5 = c$. It follows that $a \times 3 + b \times 2 = a \times 5 + b \times 5$ and then that $2 \times a = -3 \times b$. Thus one solution is $a = 3$ and $b = -2$. Plugging these values into the original equation gives $3x - 2y = c$. Replacing x and y by 3 and 2 gives $c = 5$. So we have the line $3x - 2y = 5$ passes through the point (3,2) and we easily check that it also passes through the point (5,5). Notice $-6x + 4y = -10$ is also an equation for this line. In fact, we can get as many different equations for this line as we wish by simply multiplying both sides of $3x - 2y = 5$ by any real number: $3\pi x - 2\pi y = 5\pi$ is also an equation for this line. Equations for lines are usually reported in the standard form $y = mx + b$; putting our equation in this form, we get $y = \frac{3}{2}x - \frac{5}{2}$. The only lines that do not have an equation of this form are the vertical lines. They all have simple equations of the form $x = c$ for some constant c. This can be confusing since the variable y does not appear at all. But, it simply means that the value of y is arbitrary. For example, $x = \frac{3}{2}$ has solutions $(\frac{3}{2}, 0)$, $(\frac{3}{2}, -\frac{1}{2})$, $(\frac{3}{2}, 73)$ and indeed $(\frac{3}{2}, y)$ for every possible choice for y.

Exercise 1.22. In each of the following two x, y-planes, we have graphed a line. Find an equation for each line.



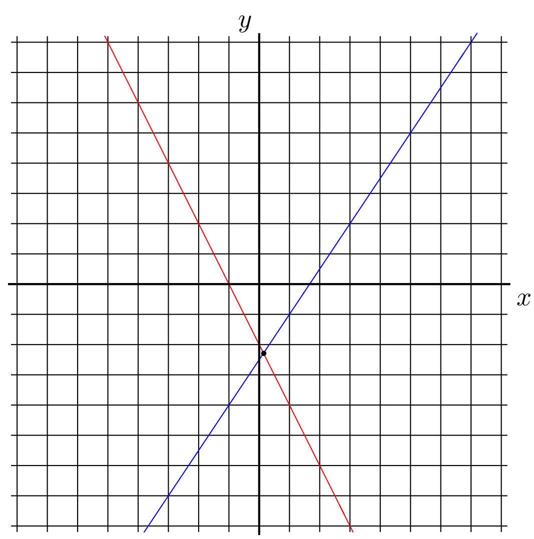


6. Systems of Equations

In the previous section solving for the coefficients a, b and c of the equation for a line resulted in solving the the pair of equations $3a + 2b = c$ and $5a + 5b = c$. We refer to such a problem as a *system* of equations and more generally a *system of algebraic sentences.* In this case, we had a system of two equations $a \times 3 + b \times 2 = c$ and $a \times 5 + b \times 5 = c$ in three unknowns a, b and c. *By a solution to a system of algebraic sentences we mean a set of values for the variables that make every algebraic sentence in the system true.* For this problem we found two solutions, $a = 3$, $b = -2$, $c = 5$ and $a = -6$, $b = 4$, $c = -10$, and we noted that there were many more.

The simplest case of systems are systems of two equations in two unknowns. For example $3x - 2y = 5$ and $2x + y = -2$. We may picture the solution to this system in the plane. Each of these individual algebraic sentences has a line as its solution set and we have graphed these below; the solution set of $3x - 2y = 5$ in blue and the solution set of $2x + y = -2$ in red. It is easy to see geometrically that there is exactly one solution; it is represented by the point where the lines cross. However, it is not easy to see exactly what that solution is. We might approximate it by taking $x = \frac{1}{6}$ and $y = -\frac{7}{3}$. Checking, we have $2(\frac{1}{6}) + (-\frac{7}{3}) = -2$; but, $3(\frac{1}{6}) - 2(-\frac{7}{3}) = \frac{31}{6} \neq 5$. We could try drawing a more accurate picture and later we will introduce some iterative techniques that would give a better approximation; but the best option is to solve the system algebraically by substitution:

- Solving $2x + y = -2$ for y gives $y = -2x - 2$.
- Substituting $-2x - 2$ for y in $3x - 2y = 5$, gives $3x - 2(-2x - 2) = 5$; which simplifies to $7x = 1$ or $x = \frac{1}{7}$
- Substituting $\frac{1}{7}$ for x in $y = -2x - 2$ then gives $y = -\frac{16}{7}$.
- The reader now can easily check that $(\frac{1}{7}, -\frac{16}{7})$ is indeed the solution.



This problem illustrates the different roles that algebra and geometry play in our development: geometry to understand or see what's going on - algebra to compute precise solutions.

Exercise 1.23. Solve each of the following systems geometrically and algebraically.

(i)

$$\begin{aligned} 2x + y &= 1 \\ x + y &= 4 \end{aligned}$$

(ii)

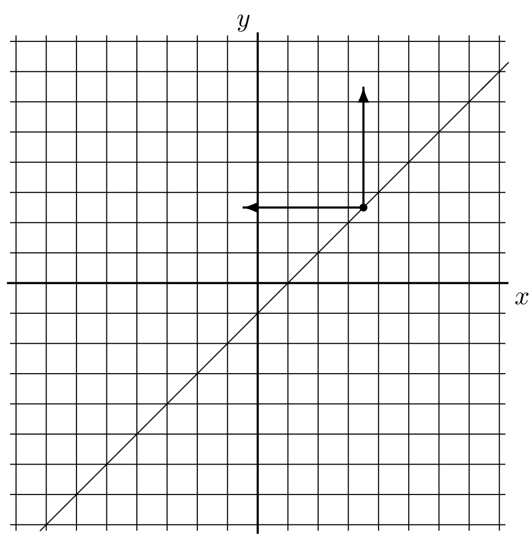
$$\begin{aligned} 2x + y &= 1 \\ 4x + 2y &= 4 \end{aligned}$$

(iii)

$$\begin{aligned} 2x + y &= 1 \\ x + y &= 4 \\ 3x - y &= 10 \end{aligned}$$

7. Inequalities

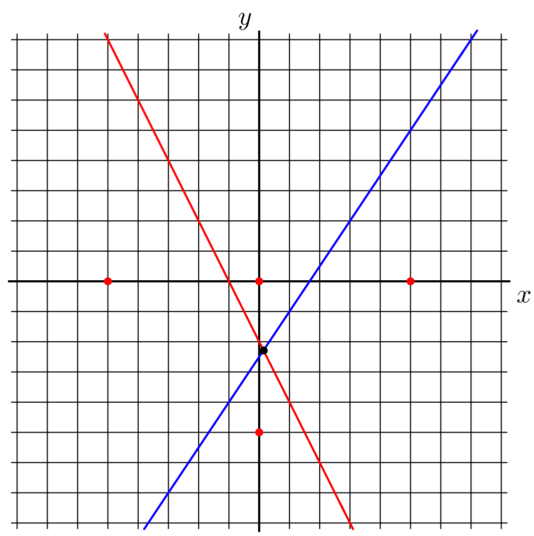
The solution sets for inequalities involving more than one variable may also be pictured on the number grid. Consider the simple inequality $x < y + 1$. The first step in picturing the solution set to this inequality, is to draw the solution set to the corresponding equality $x = y + 1$:



We have also pictured a point on this line of equality, $x = y + 1$. Note that if we increase the value of y by any amount, however small, we get a solution to our inequality. Similarly, if we decrease the value of x by any amount, however small, we again get a solution to our inequality. since this same argument applies to every point on the line, the solution set to our inequality includes all points to the left and above the line. In fact, this is the solution set. To see this, we use a similar argument identifying the points to the right and below the line as those that satisfy the inequality $x > y + 1$.

We can also visualize solutions to systems of inequalities. Consider $3x - 2y < 5$ and $2x + y > -2$. We have just graphed the two lines of the corresponding equalities and we reproduce their graphs below.

The solutions to $3x - 2y < 5$ is the set of all of the points on one side of the blue line and the solutions to $2x + y > -2$ is the set of all of the points on one side of the red line. Hence the solutions to the system is the set of all point in one of the four wedge shaped regions defined by these two lines. Which one? One easy way to find out is to try a point in each region. The natural first point to check is the origin $(0, 0)$ and it works ($0 < 5$ and $0 > -2$); a lucky first guess. It is no longer necessary, but let's check a point in each of the other regions. The point $(-5, 0)$ is in the left hand region and $-15 < 5$ but $-10 \not> -2$; the point $(5, 0)$ is in the right hand region and here $15 \not< 5$ while $10 > -2$; finally, the point $(0, -5)$ is in the bottom region, here $10 \not< 5$ and $-5 \not> -2$.



EXERCISE 1.24. Solve each of the following systems geometrically.

(i)

$$\begin{aligned} 2x + y &> 1 \\ x + y &< 4 \end{aligned}$$

(ii)

$$\begin{aligned} 2x + y &> 1 \\ 4x + 2y &< 4 \end{aligned}$$

(iii)

$$\begin{aligned} 2x + y &> 1 \\ x + y &< 4 \\ 3x - y &< 10 \end{aligned}$$

8. Functions

The variables in algebraic sentences that arise from application have specific meanings. Consider the following three algebraic sentences:

$$\begin{aligned} C &= 20Q + 2000 \\ P &= 100 - .2Q \\ p &= PQ - C = PQ - 20Q - 2000 \end{aligned}$$

In this system, Q the quantity of widgets produced and sold by the XYZ Widget Company. C is the cost of producing Q widgets. P is the sale price when Q widgets are sold. Finally, p is the profit when Q widgets are produced and sold. The first equation, called the *cost equation,* tells us that there is an *initial cost* or *fixed cost* of \$2000 to startup production and then an additional \$20 per widget produced. The \$20 for each widget is called the *marginal cost.* The second equation, called the *demand equation,* is easier to understand if we solve for Q: $Q = 500 - 5P$. Here we see that, as the price goes up the number of widgets that you will sell goes down. In particular if $P = 0$, that is if you are giving the widgets away, there is a demand for 500, on the other hand, no one would buy widgets priced at \$100 or more. Finally, we have the *profit equation* It is simply PQ, the amount of money received from sales, minus C, the cost of producing that quantity.

Let's consider a specific example. Suppose that the company produces 300 widgets. The total cost to the company is $C = 20 \times 300 + 2000 = 8000$ dollars and the price at which these 300 widgets will sell is $P = 100 - .2 \times 300 = 100 - 60 = 40$. dollars per widget. So the profit to the company is $p = 40 \times 300 - 8000 = 12000 - 8000 = 4000$ dollars. On the other hand, if the company produces and sells 400 widgets, they loose money: the cost is $20 \times 400 + 2000 = 10000$ dollars; the price is $100 - .2 \times 400 = 20$ dollars per widget; the profit is $20 \times 400 - 10000 = -2000$ for a \$2,000 loss.

There are obvious questions to ask. What is the range of profitability? Produce too few or too many and you loose money. What are the cutoffs? At what level of production will profits be maximum? To answer these question we need a better framework in which to interpret these equations. We observed in our examples that, once the variable Q had been set, values of all of the other variables are forced. We say that cost C, price P and profit p are all *functions* of the quantity Q. In function notation, we would write:

$$\begin{aligned} C(Q) &= 20Q + 2000 \\ P(Q) &= 100 - .2Q \\ p(Q) &= P(Q) \times Q - C(Q) \end{aligned}$$

For each function, we say that Q is the *independent* variable and C, P or p the *dependent* variable. We may rewrite $p(Q)$ in terms of Q alone by replacing $P(Q)$ and $C(Q)$ by the right hand sides of the corresponding equations:

$p(Q) = (100 - .2Q) \times Q - (20Q + 2000) = 100Q - .2Q^2 - 20Q - 2000 = -.2Q^2 + 80Q - 2000$

Writing or examples in function notation, we have:

$C(300) = 8000$, $P(300) = 40$, $p(300) = 4000$, all in dollars, and

$C(400) = 10000$, $P(400) = 20$, $p(400) = -2000$, again in dollars.

In working problems that arise from applications there is always the question of keeping track of "dimensions." Specifically, should we have maintained the dollar signs throughout the computations or is it OK to ignore them and only add them at the end? For example we could have written

$$p = \$40 \times 300 - \$8000 = \$12000 - \$8000 = \$4000$$

We can be even more precise and write

$$p = 40(dollars\ per\ widget) \times 300(widgets) - 8000(dollars) = 4000(dollars)$$

or

$$p = 40\left(\frac{dollars}{widget}\right) \times 300(widgets) - 8000(dollars) = 4000(dollars).$$

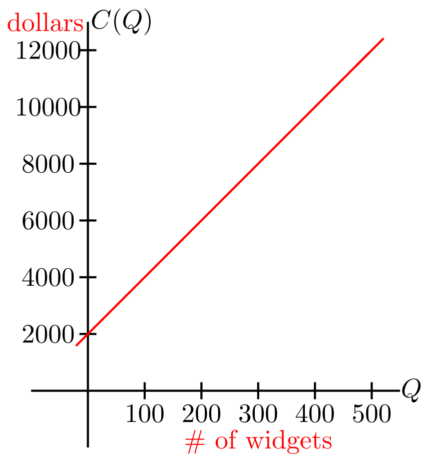
The safe way is to always include the *dimensions*, as they are called. However, this can be very cumbersome and the natural tendency is to leave them out. A good compromise is to leave them out but then go over the equation and make sure that you could put them in if required. We will discuss this issue further when working with more complicated examples. But another simple example we are all familiar with is "mpg" miles per gallon. Suppose that you buy a new hybrid that gets an average of 42mpg. and you fill the tank with 18.5 gallons of gas; how far will you get before you run out of gas? $42\frac{miles}{gallons} \times 18.5gallons = 777miles$. The fuel efficiency figure was computed by driving the car over a standard route of 1100 miles and then dividing by the amount of gas consumed 26.19 gallons: $\frac{1100miles}{26.19gallons} = 42\frac{miles}{gallons} = 42mpg$. If we invert miles per gallon, we get gallons per mile: $\frac{1}{42mpg} = .0238\frac{gallons}{miles}$. So for a 750 mile trip you can expect to use $0238\frac{gallons}{miles} \times 750miles = 17.86gallons$ of gasoline.

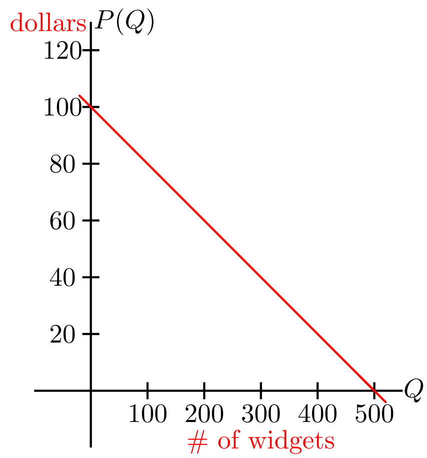
Exercise 1.25. Building on this last discussion assume that gasoline costs \$4.77 per gallon. Make the following computations keping track of the dimensions.

(i) The expected cost of a 654 mile trip.
(ii) The average cost per mile to drive you car.
(iii) The distance you could go on \$50.

Returning to our earlier business application, we review the *graphing* of functions in the coordinate plane. We assign the horizontal axis to the independent variable and the vertical axis to the dependent variable. The graph of the function $f(x)$ is then the set of all pairs of the form $(x, f(x))$. In the case of our example, the graphs of $C(Q)$ and $P(Q)$ are straight lines. The profit function is a quadratic function and its graph is a parabola that opens downward. The graph is also a good place to include dimensions making it clear that the independent variable is the number of widgets produced and the dependent variables take on dollar amounts.

In Chapter 3 we will discuss quadratic equations in detail. We will return to this problem then and answer the two questions posed above: What are the bounds of the range of profitability, that is where does the graph cross the Q-axis? and For what value of Q will profit be at its maximum and what is that maximum?


dollars
C(Q)
12000
10000
8000
6000
4000
2000
Q
100 200 300 400 500
# of widgets


dollars
P(Q)
120
100
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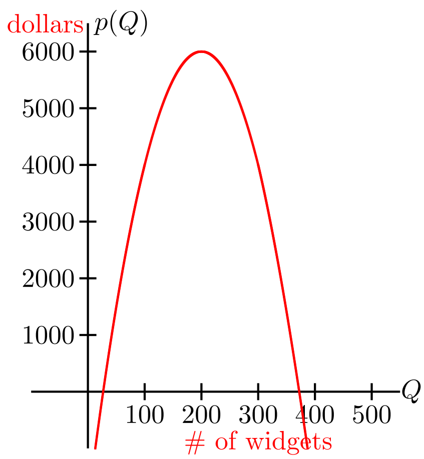
60
40
20
Q
100 200 300 400 500
# of widgets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# of widgets

CHAPTER 2

Tactical Algebra

1. The Rules of Algebra

Like any game, algebra must be played by the rules. Therefore, as with any other game, you must know the rules. Remember back to the last time you tried to learn a new game. Trying to play with the rule book in one hand is not much fun. Only after you have internalized the rules can you begin to master the game. One nice thing about algebra is that the rules are easy to understand and remember. In fact, most of us learn algebra just like we learn most games: we watch it being played by others and learn by example. But, in the end, to master any game you must eventually sit down and read the rule book. So here is the rule book for the game of algebra.

The game pieces are of three types: numbers, variables, and operations. The numbers can be any of the number sets with which we are familiar: the *natural or counting numbers* (0,1,2,...), the *integers* (...-2,-1,0,1,2,...), the *rational numbers* and the real numbers; or even more complicated number systems like the complex numbers. One usually starts learning algebra with the simplest of these systems, the natural numbers. But soon it becomes clear that the problems that we can solve with this number system is very restricted; so we soon begin to include negative numbers (moving to the integers) then fractions (moving to the rational numbers). We also start learning to play the game without the variables. In that case, the rules we are about to give are often called the *rules of arithmetic*.

The variables are usually letters from our alphabet and they are best thought of as numbers whose identity we have not yet discovered or, keeping with the game analogy, wild cards in a card game. They are sometimes referred to as the mathematical version of pronouns like "he"and "she"that can stand in for any number. We use this interpretation to state our rules in terms of variables which then hold for any numbers replacing those variables.

The rules of this game of algebra are all about combining and manipulating variables and numbers using two basic operations: addition, "+", and multiplication, "×". Later we will define the shortcut operations of substraction,"−", and division,"÷"; and some advanced operations like squaring, "$()^2$", and taking square roots "$\sqrt{\ }$". Just like points and lines in Euclidean geometry are "undefined terms,"addition and multiplication are the undefined terms in the set of rules for algebra. We will assume that, given any two numbers or variables, x and y, in our number system, we may compute their sum $x + y$ and their product $x \times y$. In Chapter 1, we intrepreted our numbers as lengths and defined addition and multiplication geometrically. In different systems they may be defined differently; but, no matter how they are defined they must satisfy the rules that we are about to list.

We will use these operations to combine numbers and variables into algebraic expressions and we will use the rules to manipulate these expressions. Finally, as with Euclid, we need to agree on the fundamental properties of the concept of *equality:*

(i) (reflexive) Each algebraic expression is equal to itself.
(ii) (symmetric) If Expression A is equal to expression B, then expression B is equal to expression A.
(iii) (transitive) If Expression A is equal to expression B and expression B is equal to expression C, then expression A is equal to expression C.
(iv) (adding equals to equals) If expression B is equal to expression A and expression C is equal to expression D, then expression $A+C$ is equal to expression $B+D$.
(v) (multiplying equals by equals) If expression B is equal to expression A and expression C is equal to expression D, then expression $A\times C$ is equal to expression $B\times D$.

Rule 1. Addition is *commutative*

$$x+y=y+x \quad \text{for all numbers and variables } x \text{ and } y.$$

Rule 2. Addition is *associative*

$$(x+y)+z=x+(y+z) \quad \text{for all numbers and variables } x,\ y \text{ and } z.$$

Exercise 2.1. Choose numerical values (integers, rational or real numbers) for the quantities (x, y, and/or z) appearing in Rules 1 and 2. Use your calculator to find the value of each side of the equation and verify that the equation is valid for the numbers that you chose.

This exercise may give us some level of confidence that addition on our number systems do satisfy these rules. But, such examples do not prove that the rules hold for all choices of numbers. However, as you will soon see, a specific example that does not satisfy a rule is sufficient to show that an operation on a number system does not satisfy a rule These two rules seem so natural to us that you may wonder why we make such a fuss over them. The reason is that there are operations that do not satisfy these rules.

Exercise 2.2. Show each of the following with a simple example.

(i) Subtraction is not commutative.
(ii) Subtraction is not associative.
(iii) Division is not commutative.
(iv) Division is not associative.

Rule 3. There is a special number called the *additive identity* and denoted by 0 with the property that

$$0+x=x \quad \text{for all numbers and variables } x.$$

Rule 4. For every number x there is another number called *its additive inverse* and denoted by $-x$ with the property that

$$x+(-x)=0.$$

EXERCISE 2.3. Find the additive inverse for each of the following numbers and variables.

(i) 7
(ii) -7
(iii) $\frac{1}{2}$
(iv) $-x$

At this point, we could give a formal definition for addition on the natural numbers, the integers, the rational numbers and the real numbers and then prove that they all satisfy Rules 1, 2 and 3 and that all but the natural numbers satisfy Rule 4. To do this would draw us into the logical foundation of these number systems and would take us away from our main purpose: review and strengthen our algebra skills. Hence, we will assume that natural numbers, the integers, the rational numbers and the real numbers all satisfy Rules 1, 2 and 3 and that all but the natural numbers satisfy Rule 4.

We now turn to multiplication. It is required to satisfy a similar set of four rules.

Rule 5. Multiplication is *commutative*

$$x \times y = y \times x \quad \text{for all numbers and variables } x \text{ and } y.$$

Rule 6. Multiplication is *associative*

$$(x \times y) \times z = x \times (y \times z) \quad \text{for all numbers and variables } x,\ y \text{ and } z.$$

Rule 7. There is a special number, 1, called the *multiplicative identity* with the property that

$$1 \times x = x \quad \text{for all numbers and variables } x.$$

Rule 8. For every number, **except 0**, there is another number called its *multiplicative inverse* and denoted by $\frac{1}{x}$, or by x^{-1}, with property that

$$x \times \frac{1}{x} x \times x^{-1} = 1.$$

In Section 4, Modifying algebraic sentences, of the last chapter we discussed the necessity for making 0 an exception to this rule. As above, we will assume that natural numbers, the integers, the rational numbers and the real numbers all satisfy Rules 5, 6 and 7 and that the rational numbers and the real numbers satisfy Rule 8.

EXERCISE 2.4. Find the multiplicative inverse for each of the following numbers and variables.

(i) 7
(ii) -7
(iii) $\dfrac{1}{2}$
(iv) $\dfrac{1}{x}$
(v) $-x$

Most of our numeric and algebraic computations involve both of these basic operations. So we must be precise about how they interact. The following rule is just what we need and all four number systems satisfy this distributivity rule:

Rule 9. Multiplication *distributes* over addition. That is,

$$x \times (y + z) = x \times y + x \times z \quad \text{for all numbers and variables } x,\ y \text{ and } z,$$

and

$$(x + y) \times z = x \times z + y \times z \quad \text{for all numbers and variables } x,\ y \text{ and } z.$$

EXERCISE 2.5. Choose numerical values for the quantities (x, y, and/or z) appearing in Rules 5, 6 and 9. Use your calculator to find the value of each side of the equation and verify that the equation is valid *for these numbers*.

There are many number systems that satisfy these nine rules beyond the the integers, rationals and reals. We have already mentioned the complex numbers. Another interesting set of examples are the finite number systems: the *integers mod* m. The set of numbers in this system are $\{0, 1, 2, \ldots, m-1\}$. Two numbers in this number system are added or multiplied as integers and, if the result is greater than $m-1$, m is subtracted from the result as many times as need to get a number in $\{0, 1, 2, \ldots, m-1\}$. More precisely $x +_m y$ and $x \times_m y$ are the remainders of $x + y$ and $x \times y$ when divided by m. Computing in these systems is called *modular arithmetic.* When m is a prime all nine of the above rules hold and one can do algebra in as usual in these number systems. But, when is not a prime, not all of the rules hold. Specifically, multiplicative inverses do not exist for some numbers in the system.

EXERCISE 2.6. Consider the integers mod 5 and then the integers mod 6:

(i) Fill out an addition table and a multiplication table.
(ii) For each non-zero number, list its additive inverse.
(iii) For each non-zero number, list its multiplicative inverse.

One of the special features of "our" number systems (the integers, rationals and reals) that distinguish them from the complex numbers and the modular numbers is the concept of *positive numbers.*

Positive Numbers. The positive numbers in each of our three number systems are the numbers that appear to the right of 0 when the number system is pictured on the number line. Positive numbers have several useful properties:

(i) The positive numbers are closed under addition (if x and y are positive then $x + y$ is positive).
(ii) The positive numbers are closed under multiplication (if x and y are positive then $x \times y$ is positive).
(iii) If x is positive then so is its multiplicative inverse.
(iv) For every number x exactly one of the following holds:
  (a) $x = 0$;
  (b) x is positive;
  (c) x is negative, that is $-x$ is positive.
(v) The positive numbers enable us to define the relations $<$ and $>$: we write $x < y$, if $y - x$ is positive and $x > y$, if $x - y$ is positive.

Next we define the shortcut operation of subtraction in terms of addition and the shortcut operation of division in terms of multiplication.

Definition of Subtraction. $x - y$ is defined to be $x + (-y)$. That is, $x - y$ is x plus the additive inverse of y.

Since addition is associative and commutative, and using the property of the additive inverse, we see that

$$(x - y) + y = (x + (-y)) + y = x + ((-y) + y) = x + 0 = x.$$

This shows that $x - y$ is the quantity that when added to y gives x.

Note that there is a subtle difference between the operation of subtraction and the operation of taking the additive inverse, though both are denoted with a minus sign. Scientific calculators distinguish these operations by two different keys: (-) for the additive inverse and $-$ for subtraction.

Exercise 2.7. Enter each of the following expressions in your calculator. Some might result in an error message. Try to predict in advance which expressions will give an error message.

(i) $7 - -5$
(ii) 7(-)(-)5
(iii) 7$-$(-)5
(iv) 7(-)$-$5
(v) 7+$-$5
(vi) 7+(-)5.

Explain each case that result in an error message.

Definition of Division. $x \div y$ is defined to be $x \times \left(\frac{1}{y}\right)$. That is, $x \div y$ is x times the multiplicative inverse of y.

Multiplication is associative and commutative, and by using the property of the multiplicative inverse, we see that

$$(x \div y) \times y = \left(x \times \left(\frac{1}{y}\right)\right) \times y = x \times \left(\left(\frac{1}{y}\right) y\right) = x \times 1 = x.$$

This shows that $x \div y$ is the quantity that when multiplied by y gives x.

Scientific calculators usually have a key for the operation of division, $\div$, and a separate key, x^{-1} for the operation of taking the multiplicative inverse. On some calculators, the multiplicative inverse, of say 5, is found using the key strokes 5$\wedge$(-)1.

Exercise 2.8. To do the following calculations on your calculator. Some might result in an error message. Try to predict in advance which expressions will not give an error message and explain in words the calculated result.

(i) $35 \div 7$
(ii) 35×7^{-1}
(iii) 35 $\div$ (-)7
(iv) 35(-) $\div$7
(v) $35 \div 7^{-1}$
(vi) $35^{-1} \div 7$

There are many shortcuts that we use when we play the game of algebra; we list several below. We are all used to using them without a thought. However, we claim to have given a complete set of rules. So, you are playing by these rules and replace $-(-x)$ by x your "opponent" might claim a foul! Unless, of course, you can explain just how the rules permit this move. Each of the shortcuts listed below can be shown to be a consequence of the nine rules listed above. Some are easy to see others are not so easy to see. We will verify a few of the more complicated ones leaving the rest as an exercise.

Useful Facts and Shortcuts: Each of the following equations is valid for all values of the variables x and y:

(i) the additive inverse of x is unique;
(ii) for $x \neq 0$, the multiplicative inverse of x is unique;
(iii) $0x = 0$;
(iv) if $xy = 0$, then either $x = 0$ or $y = 0$;
(v) $-x = (-1) \times x$;
(vi) $-(-x) = x$;
(vii) $(-x) \times (-y) = x \times y$;
(viii) $(-x) \times y = x \times (-y) = -(x \times y)$;
(ix) $\frac{1}{-y} = -\frac{1}{y}$;
(x) $\frac{-x}{-y} = \frac{x}{y}$ $(y \neq 0)$;
(xi) $\frac{-x}{y} = \frac{x}{-y} = -\frac{x}{y}$, $(y \neq 0)$;
(xii) $w \div \frac{x}{y} = w \times \frac{y}{x}$, $(x, y \neq 0)$.

PROOF. (i) Actually, Rule 4 does seem to indicate that the additive inverse of x is unique by giving it the symbol $-x$. But, just suppose that there was another additive inverse for x, call it w. That is, $w + x = x + w = 0$ too. Now consider the following string of equalities:

$$w = w + 0 = w + (x + (-x)) = (w + x) + (-x) = (0) + (-x) = -x$$

So we were justified in denoting *the* additive inverse of x by $-x$. The proof that the multiplicative inverse is unique is similar: assume $x \neq 0$ and $wx = 1$, then

$$w = w1 = w(xx^{-1}) = (wx)x^{-1} = 1x^{-1} = x^{-1}$$

To verify (iii), let $w = 0x$. Then $w + w = 0x + 0x = (0 + 0)x = 0x = w$. So $w + w = w$; add $-w$ to both sides to get:

$$w = w + w + (-w) = w + (-w) = 0.$$

Turning to (iv), assume that $xy = 0$. If $x = 0$ there is nothing to prove; so assume that $x \neq 0$. We must show that, in this case, $y = 0$.

$$y = 1y = (\frac{1}{x}x)y = \frac{1}{x}(xy) = \frac{1}{x}0 = 0.$$

The following string of equalities verfies (v): $-1x + x = -1x + 1x = (-1 + 1)x = 0x$. $\square$

EXERCISE 2.9. Verify the remaining seven shortcuts.

From now on we will freely use these shortcuts in our computations.

2. Expanding, Simplifying and Solving

Two of the basic tactics for dealing with expressions and equations are *expanding* and *collecting like terms;* combining these two operations is often called *simplifying.* Consider the following equation:

$$(7+x)(y-4)=3x-7$$

Suppose that we wish to solve for the variable x. Our strategy is:

(i) first, expand the products;
(ii) next, move all terms involving x to one side and all terms not involving x to the other;
(iii) then, factor out the x and
(iv) isolate it by dividing through by its coefficient.

Start by expanding the lefthand side using the distributive rule:

$$\begin{aligned} (7+x)(y-4) &= 3x-7 \\ (7+x)y+(7+x)(-4) &= 3x-7 \\ 7y+xy+7(-4)+x(-4) &= 3x-7 \\ 7y+xy-28-4x &= 3x-7 \end{aligned}$$

Next we collect all of the terms involving x on the left and all of the remaining terms on the right:

$$\begin{aligned} 7y+xy-28-4x &= 3x-7 \\ xy-4x-3x &= 28-7-7y \\ xy-7x &= 21-7y \end{aligned}$$

Factoring (using the distributive rule in reverse) x out of the left hand expression and 7 out of the righthand expression gives:

$$\begin{aligned} xy-7x &= 21-7y \\ (y-7)x &= 7(3-y) \end{aligned}$$

Finally, dividing both sides by $(y-7)$, that is multiplying both sides by the multiplicative inverse of $(y-7)$, $\dfrac{1}{(y-7)}$, we get:

$$\begin{aligned} (y-7)x &= 7(3-y) \\ x &= \frac{7(3-y)}{y-7} \end{aligned}$$

Until this last step, we made no restrictions on the values that the variable y could take on. However, $(y-7)$ has an inverse only if it is a number different from 0. So to carry out this last step, we must require that $(y-7)\neq 0$ or $y\neq 7$.

EXERCISE 2.10. For $y=0,4$ and $\frac{1}{2}$, substitute this value for y then solve the original equation, $(7+x)(y-4)=3x-7$ for x and compare those result with the answers you get by substituting the value for y in $x=\dfrac{7(3-y)}{y-7}$. Then evaluate both sides of the original equation for the case $y=7$ and explain why we cannot solve for x.

In the above example, we explicitly stated when we use rules 8 and 9, the existence of multiplicative inverses and the distributivity rule. But we use several of the other rules, some several times. Work through that example and, for each step, identify all rules that were used.

While the above example was designed to illustrate the power of algebra by involving most of the rules of algebra, the best way to learn the rules and how to use them is to start with a simple example using only one or two rules at a time.

Exercise 2.11. In each case, carry out the instructions and list each rule or rules that you used.

(i) Simplify $(x - y) + (3y - x)$;
(ii) Solve $4(3a + 1) = 8$ for a;
(iii) Solve $17m + w = 8$ for w;
(iv) Solve $17m + 3w = 8$ for w;
(v) Solve $17a + b = 8$ for a;
(vi) Simplify $4 \times x \times 3 - 5x + 3(y - x)$;
(vii) Solve $4 \times x \times 3 - 5x + 3(y - x) = 0$ for x;
(viii) Solve $3(x - y + 4) = 6$ for x;
(ix) Expand $(A + b - 5)(4 - P + x)$;
(x) Solve $(A + b - 5)(4 - P + x) = 10$ for the term AP;
(xi) Solve $(7 + x)(y - 4) = 3x - 7$ for y.

3. Substitution

Another very useful technique for solving equations and simplifying expressions is *substitution*, replacing a single variable by an expression. We illustrate with a very simple example:

Solve the system of equations $x + y = 3$ and $x - y = 2$.

We start by solving the first equation for x to get $x = 3 - y$. Then we substitute $3 - y$ for the x in the second equation to get $(3 - y) - y = 2$ or $3 - 2y = 2$. Solving this equation for y gives $y = \frac{1}{2}$. And finally, substituting $\frac{1}{2}$ for y in $x = 3 - y$ gives $x = 3 - \frac{1}{2} = \frac{5}{2}$. Checking we have $\frac{5}{2} + \frac{1}{2} = \frac{6}{2} = 3$ and $\frac{5}{2} - \frac{1}{2} = \frac{4}{2} = 2$.

One of the powers that algebra gives you is the power to generalize. Stepping back and looking at this last problem we see that we have been given the sum and difference of two numbers and asked to find the two numbers. We can solve all such problems at one time:

Given that $x + y = S$ and $x - y = D$, find x and y in terms of the sum S and the difference D. To do this we carry out exactly the same steps as we did when S and D were the numbers 3 and 2:

Solving for x in the first equation, $x = S - y$.

So $(S - y) - y = D$, $S - D = 2y$ and $\dfrac{S - D}{2} = y$.

Finally, $x = S - \dfrac{S - D}{2} = \dfrac{2S}{2} - \dfrac{S - D}{2} = \dfrac{S + D}{2}$.

We state the general solution to this problem:

If $x + y = S$ *and* $x - y = D$ *then* $x = \dfrac{S + D}{2}$ *and* $y = \dfrac{S - D}{2}$.

So for our first example, we can simply write down the solution: $x = \dfrac{3 + 2}{2} = \dfrac{5}{2}$ and $x = \dfrac{3 - 2}{2} = \dfrac{1}{2}$.

One of the wonderful features of algebra is that usually there are several different ways to achieve the same result. To solve this system, $x + y = S$ and $x - y = D$, we could take advantage of the properties of equalities: adding them, we have

$(x + y) + (x - y) = S + D$ or simply $2x = S + D$ and $x = \dfrac{S + D}{2}$. Subtracting them, we have $(x + y) - (x - y) = S - D$ or simply $2y = S - D$ and $y = \dfrac{S - D}{2}$.

EXERCISE 2.12. Solve each of the following systems of two equations in two unknowns by substitution. Check your answers by graphing the two lines.

(i) $7x - 5y = 26$, $2x - 3y = 3$;
(ii) $21x - 16y = 11$, $15x + 4y = 6$;
(iii) $21x - 15y = \frac{1}{4}$, $-14x + 10y = \frac{1}{6}$.

Substitution can also be used to systematically solve systems of three linear equations in three unknowns, four equations in four unknowns and so on. The strategy is straight forward:

(i) Solve the first equation for one of its variables in terms of all of the others.
(ii) Replace this variable in the remaining equations by its equal expression involving the other variables.
(iii) You now have one equation solved for your selected variable and a system equations with one fewer equation and one fewer variable. Solve it for the remaining variables and use their values to determine the value of your selected variable.
(iv) Of course, this smaller system can be solved using this same strategy.

To illustrate we will solve the following system of four equations in four variables:

$$\begin{array}{rcrcrcrcr} 2w & - & 3x & + & y & + & z & = & 23 \\ w & + & x & + & y & + & z & = & 10 \\ 5w & - & 4x & + & 3y & - & 2z & = & 3 \\ & & 5x & - & 3y & + & z & = & -11 \end{array}$$

Solving the first equation for z, we have:

$z = 23 - 2w + 3x - y$

Substituting $(23 - 2w + 3x - y)$ for z in the second equation

$w + x + y + (23 - 2w + 3x - y) = 10$

and collecting like terms, gives:

$-w + 4x = -13.$

Then making this substitution in the third and forth equations yields the following system of three equations in three variables:

$$\begin{array}{rcrcrcr} -w & + & 4x & & & = & -13 \\ 9w & - & 10x & + & 5y & = & 49 \\ -2w & + & 8x & - & 4y & = & -34 \end{array}$$

The first step in solving this system is solving the first equation for w:

$w = 4x + 13$

Substituting $(4x + 13)$ for w in the second and third equations, we get:

$$\begin{array}{rcrcr} 26x & + & 5y & = & -68 \\ & - & 4y & = & -8 \end{array}$$

Solving this system we have $y = 2$ and $x = -3$. Then, putting these values into $w = 4x + 13$ give $w = 1$. Finally, evaluating $z = 23 - 2w + 3x - y$ gives $z = 10$. So our solution is

$$w = 1,\ x = -3,\ y = 2,\ z = 10.$$

While this method will always find the solution of a system of equations, if there is one, it involves many, sometimes unpleasant, computations. Furthermore, there are more efficient methods for solving systems of linear equations. However, we won't pursue this any further since our interest here is to simply demonstrate the power of the technique of substitution.

Exercise 2.13. Use the method of substitution to solve the following two systems of linear equations.

$$\text{(i)}\quad \begin{array}{rcrcrcr} 5x & + & 6y & - & 2z & = & 4 \\ 5x & + & 9y & + & 14z & = & 9 \\ -x & - & 15y & + & 14z & = & -2 \end{array}$$

$$\text{(ii)}\quad \begin{array}{rcrcrcrcr} x & - & y & + & z & - & w & = & 0 \\ x & + & y & - & z & + & w & = & 2 \\ x & - & y & - & z & - & w & = & -2 \\ x & + & y & + & z & + & w & = & 4 \end{array}$$

Reconsider the equation $(7+x)(y-4) = 3x-7$ from the previous section and assume that $3x + y = 6$ also holds. Solving the second equation for y ($y = 6 - 3x$) and substituting $(6 - 3x)$ for y in the first equation we get

$$(7+x)(6-3x+4) = 3x-7.$$

Expanding the left side and simplifying:

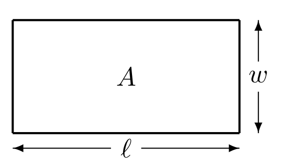
$$\begin{aligned} (7+x)(6-3x+4) = 3x-7 \\ (7+x)(10-3x) = 3x - 770 - 21x + 10x - 3x^2 &= 3x-7 \\ -3x^2 - 11x + 70 &= 3x-7 \\ -3x^2 - 14x + 77 &= 0 \\ 3x^2 + 14x - 77 &= 0 \end{aligned}$$

The result is an equation in one variable. However, it is a quadratic equation. In the next chapter we will consider quadratic equations and how to solve them. At that time we will return to this problem.

Exercise 2.14. Solve this same system for a single quadratic equation in y.

4. Formulas

The techniques that we developed to simplify algebraic sentences were applied only to very simple sentences. However, the formal rules of algebra enable us to work with more complicated algebraic sentences that arise from applications. The term *formula* is often applied to algebraic sentences that come from real world applications. For example, the formulas for the perimeter and area of a rectangle: $P = 2\ell + 2w$ and $A = \ell \times w$ (or $A = \ell w$), where ℓ denotes the length and w the width of the rectangle.



Using our techniques, we may *solve* these formulas for any one of the variables. For example, we may solve the perimeter formula for ℓ. Start with $P = 2\ell + 2w$ and subtract $2w$ from both sides to get $P - 2w = 2\ell$. Next divide both sides by 2 and change the order to put the variable for which we are solving first:

$$\ell = \frac{P-2w}{2} = \frac{1}{2}(P-2w) = \frac{1}{2}P - w.$$

Similarly, we may solve the area formula for w to get a formula for the width of a rectangle that has area A and length ℓ. Start with $A = \ell w$. Since ℓ is always positive, we may divide both sides by ℓ (or multiply each side by $\frac{1}{\ell}$) to get: $\frac{A}{\ell} = w$ or $w = \frac{A}{\ell}$.

EXERCISE 2.15. A square with side length s is just the special case of a rectangle where $\ell = w = s$

(i) Give the formula for the perimeter P of a square in terms of s.
(ii) Solve that formula for s, that is, give the formula for the side length of a square that has perimeter P.
(iii) Give the formula for the area A of a square in terms of s.
(iv) Give the formula for the area A of a square in terms of its perimeter P.
(v) Use your formula to find each square that has the number of units of measure in its perimeter equal to the number of units squared in its area. [Be careful, there are two solutions; find them both!]
(vi) Can you describe the squares that have the number of units of measure in their perimeters greater than (less than) the number of units squared in their areas?

Now consider the general rectangle and assume that the number of units of measure in its perimeter equal to the number of units squared in its area. What can we say about ℓ and w? We have

$$\ell w = 2\ell + 2w \quad \textit{or} \quad \ell w - 2\ell - 2w = 0$$

One of the standard "tricks" used by those who play algebra often is *completing the product*. We may recognize that the left hand side consists of three of the terms of the expansion of the product $(\ell - 2)(w - 2) = \ell w - 2\ell - 2w + 4$. So if we add 4 to both sides of $\ell w - 2\ell - 2w = 0$, we get $\ell w - 2\ell - 2w + 4 = 4$ or $(\ell - 2)(w - 2) = 4$. If we require ℓ and w to be whole numbers $(\ell - 2)(w - 2)$must be a factorization of 4: $(\ell - 2) = 1$ and $(w - 2) = 4$ or $(\ell - 2) = 2$ and $(w - 2) = 2$ or $(\ell - 2) = 4$ and $(w - 2) = 1$. So the only integer sided rectangles with the number of units of measure in its perimeter equal to the number of units squared in its area are the 4×4 square and the 3×6 rectangle.

EXERCISE 2.16. Consider the rectangle pictured above.

(i) Attack the above problem by directly solving $\ell w = 2\ell + 2w$ for ℓ in terms of w and again find all integer solutions.
(ii) Use your formulas for the area and perimeter of a rectangle to find each rectangle that has the number of units of measure in its perimeter equal to twice the number of units squared in its area and has ℓ and w whole numbers. What if we permit fractional dimensions?
(iii) Use your formulas for the area and perimeter of a rectangle to find each rectangle that has twice the number of units of measure in its perimeter as the number of units squared in its area and has ℓ and w whole numbers. What if we permit fractional dimensions?

Thinking of our two equations $P = 2\ell + 2w$ and $A = \ell w$ as a system we may "solve" this system for any two variables in terms of the remaining two. They are already solved for P and A. To solve for A and w, we start with the equations $w = \frac{1}{2}P - \ell$ and $\ell = \frac{1}{2}P - w$ derived above. Substituting $\frac{1}{2}P - w$ for ℓ in $A = \ell w$ gives $A = \frac{1}{2}Pw - w \times w = \frac{1}{2}Pw - w^2$. So our new system is:

$A = \frac{1}{2}Pw - w^2$ and $w = \frac{1}{2}P - \ell$.

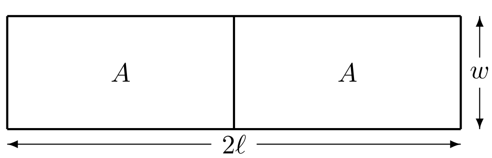
EXERCISE 2.17. Solve the system $P = 2\ell + 2w$ and $A = \ell w$ for P and ℓ

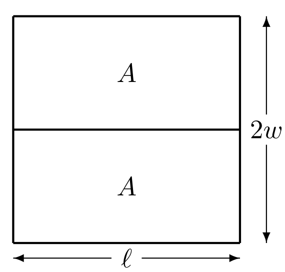
If we attempt to solve this system for ℓ and w in terms of A and P, something strange happens. Start with the equations $w = \frac{1}{2}P - \ell$ and $\ell = \frac{A}{w}$, derived above and substitute $\frac{A}{w}$ for ℓ in the first equation to get $w = \frac{1}{2}P - \frac{A}{w}$. Then multiplying both sides by w and moving all terms to the left side, we get: $w^2 - \frac{P}{2}w + A = 0$. Similarly, solving for ℓ in terms of P and A yields $\ell^2 - \frac{P}{2}\ell + A = 0$. So ℓ and w are both solutions to the same quadratic equation $x^2 - \frac{P}{2}x + A = 0$. Evidently this equation has two solutions. At this point, we cannot solve a quadratic equation; so we will return to this problem in the chapter on quadratic equations.

We close this section by using the area formula to illustrate a few of the formal rules of algebra. Two of the simplest rules for multiplication are commutativity and associativity: we may change the order of multiplying numbers and variables without changing the over-all product. For example,

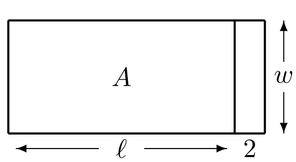
$$2A = 2(\ell w) = (2\ell)w = \ell(2w).$$

but each of the terms on the right has a different interpretation: $2A$ or $2(\ell w)$ simply indicates the number equal to twice the area of of the rectangle with length ℓ and width w; $(2\ell)w$, on the other hand, represents the area of the rectangle with length 2ℓ and width w while $\ell(2w)$ represents the area of the rectangle with length ℓ and width $2w$. One easily sees that they all represent areas of equal value.



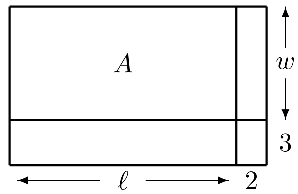


Next we illustrate distributivity. Suppose that we extend the length of the rectangle by 2 units. How will the area change? Let N denote the area of the new rectangle and A the area of the original rectangle. By distributivity $N = (\ell+2)w = \ell w + 2w = A + 2w$:

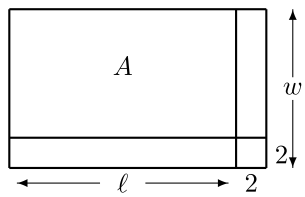


So the area added by extending the length by 2 units is $2w$, the area of a $2 \times w$ rectangle.

What if we extend the length by 2 and the width by 3? by distributivity we have: $N = (\ell + 2)(w + 3) = \ell w + 3\ell + 2w + 6$. Geometrically this means that the new rectangle should decompose into four rectangles with areas $A = \ell w$, 3ℓ, $2w$ and 6:



EXERCISE 2.18. Consider the $\ell \times w$ with $\ell > w > 2$ and alter it by increasing its length by 2 and decreasing its width by 2. How are the perimeter and area altered? Are they increased or decreased? Identify the old and new rectangles in the following diagram.



EXERCISE 2.19. Consider the $\ell \times w$ with $\ell > w > 0$ and alter it by decreasing its length by x and increasing its width by x. How are the perimeter and area altered? Are they increased or decreased? Draw a diagram and Identify the old and new rectangles. What value of x would make the area largest?

5. Working with Exponents

In Chapter 1, we introduced exponential notation x^2 for $x \times x$, x^3 for $x \times x \times x$ and in general x^n for n copies of x multiplied together. In addition to being a convenient shorthand notation, some computations become simpler in exponential notation. Consider $x^5 \times x^4$. By direct count this is 9 copies of X multiplied together. In general, $x^n \times x^m = x^{n+m}$, again by direct count. Also by a direct count $(x^4)^3 = x^4 \times x^4 \times x^4 = x^{12} = x^{4\times 3}$ and, in general, $(x^m)^n = x^{mn}$. So our two basic rules for working with exponents are:

Rules for exponents.

(i) $x^n \times x^m = x^{n+m}$;
(ii) $(x^m)^n = x^{mn}$.

EXERCISE 2.20. Using the above rules, commutativity and associativity of multiplication and distributivity, expand and simplify each of the following:

(i) $x^3y^4 \times y^5x^6$
(ii) $(x^3y^4)^2$
(iii) $(x^3y^4)^3$
(iv) $x^3y^4(x^3y^5 + y^4x^6)$

Rather than expanding that last expression, we may choose to simplify by pulling out front as many powers of x and y as possible. Simplify each of the remaining expressions in this way.

(v) $x^3y^4(x^3y^5 + y^4x^6)$
(vi) $3x^{15}y^{24}(6x^{13}y^{14} - 9y^5x^6)$

Now we wish to extend our definition of x^n to values for n other than 1, 2, …, and we want to do so in such a manner that the above rules of exponents still hold.

Let's start with x^0. We insist that $x^0 \times x^n = x^{0+n} = x^n$. Hence we must define x^0 to be 1 for every variable or number x with one exception $x = 0$. We treat the expression 0^0 as undefined. Next consider x^{-1}. We have already defined this to be $\frac{1}{x}$, again with the proviso thar $x \neq 0$. Is this definition consistent with our rules for exponents? One easily checks that $x^{-1} \times x^n = \frac{1}{x}x^n = \frac{1}{x}xx^{n-1} = x^{n-1}$ as the rules predict. We can interpret x^{-2} in two ways:

$$x^{-2} = x^{-1} \times x^{-1} = \frac{1}{x}\frac{1}{x} = \frac{1}{x^2} \quad \textit{or} \quad x^{-2} = (x^2)^{-1} = \frac{1}{x^2}.$$

In general, defining $x^{-n} = \frac{1}{x^n}$ whenever $x \neq 0$, is consistent with all of the rules.

Exercise 2.21. Assuming $x, y \neq 0$, simplify each of the following expressions.

(i) $x^{-3}y^4 \times y^{-5}x^6$
(ii) $(x^{-3}y^4)^{-2}$
(iii) $\frac{(x^3y^4)^2}{y^{-5}x^6}$
(iv) $x^{-3}y^4(x^3y^{-5} + y^4x^6)$

Rewrite each of the answers to the above in two ways: First, with out fractions using positive and negative exponents as needed; second with only positive exponents using fractions as needed.

We can extend the definition of exponents even further. Consider $2^{\frac{1}{2}}$. To be consistent with the rules we must insist that $(2^{\frac{1}{2}})^2 = 2^1 = 2$. So $2^{\frac{1}{2}}$ must equal $\sqrt{2}$. Actually, there are two possible choices for $2^{\frac{1}{2}}$: 2 and -2. In general $x^{\frac{1}{2}}$, will only be defined when $x \geq 0$ and then it will be defined to be the positive square root $\sqrt{x}$. For fractional exponents, we will adopt the convention that whenever there is a choice, we will always choose the positive interpretation. Sometimes there is no choice: $(-8)^{\frac{1}{3}} = -2$ not 2.

Exercise 2.22. Assuming $x, y > 0$, simplify each of the following expressions.

(i) $x^{-3}y^{\frac{1}{4}} \times y^{-8}x^{\frac{1}{6}}$
(ii) $(x^{-3}y^4)^{-\frac{1}{2}}$
(iii) $\frac{(x^3y^4)^{\frac{1}{2}}}{y^{-5}x^6}$
(iv) $x^{-\frac{1}{3}}y^{\frac{1}{4}}(x^3y^{-5} + y^4x^6)$

By a *rational expression* we mean a quotient of simpler algebraic expressions; for example, $\frac{(x^3y^4)}{y^{-5}x^6}$. We add or subtract rational expressions using the same technique that we use to add or subtract rational numbers: *rewrite the expressions with a common denominator.*

$$\frac{1}{x} + \frac{1}{y} = \frac{y}{xy} + \frac{x}{xy} = \frac{x+y}{xy}.$$

Exercise 2.23. Write each of the following sums as a single fraction:

(i) $\frac{1}{x} - \frac{1}{y}$;
(ii) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$;
(iii) $\frac{1}{xy} + \frac{1}{yz} + \frac{1}{xz}$;
(iv) $\frac{1}{xy^2} - \frac{1}{x^3y}$;
(v) $\frac{1}{1-x} - \frac{1}{1+x}$;
(vi) $\frac{1}{1-x} - \frac{1}{1+x} + \frac{1}{1-x^2}$.

There are some important applications of algebra that require us to reverse this process, that is, to take a rational expression and rewrite it as the sum of simpler rational expressions. For example, the expression $\frac{x}{(x-2)(x+1)}$ can be rewritten in the form $\frac{a}{x-2} + \frac{b}{x+1}$. The problem is to find the a and b. To do this we combine $\frac{a}{x-2}$ $\frac{b}{x+1}$ over a common denominator, set the numerator equal to x and solve for a and b:

$$\frac{a}{x-2} + \frac{b}{x+1} = \frac{ax+a}{(x-2)(x+1)} + \frac{bx-2b}{(x-2)(x+1)} = \frac{(a+b)x+(a-2b)}{(x-2)(x+1)}.$$

Setting $(a+b)x + (a-2b) = x$ gives us the system of equations $a+b=1$ and $a-2b=0$. So $a=2b$ and substituting $2b$ for a in the first equation gives $b=\frac{1}{3}$ and $a=\frac{2}{3}$. We have:

$$\frac{x}{(x-2)(x+1)} = \frac{\frac{2}{3}}{x-2} + \frac{\frac{1}{3}}{x+1} \quad \textit{or} \quad \frac{2}{3(x-2)} + \frac{1}{3(x+1)}.$$

The result of applying this process is called a resolution into *partial fractions.*

Exercise 2.24. Resolve each of the following rational expressions into partial fractions:

(i) $\frac{2}{(x-2)(x+1)}$;
(ii) $\frac{x-1}{(x-2)(x+1)}$;
(iii) $\frac{x-1}{(x-2)(x+1)(2x-1)}$;
(iv) $\frac{x^2-1}{(x-2)(x^2+1)}$.

Exercise 2.25. Consider the following "proof" that 1=2:
Let $a = b$, that is these variables must take on the same value, but that otherwise that common value is unrestricted. We may multiply both sides by a to get:
$a^2 = ab$.
We may then subtract b^2 from both sides to get:
$a^2 - b^2 = ab - b^2$
Now factor both sides
$(a-b)(a+b) = (a-b)b$
and divide both sides by $(a-b)$ to get:
$a+b=b$.
At no time in this sequence of operations did we constrain the common value of a and b; so we may set $a=b=1$ and conclude:
$2=1$!
Explain the flaw in this derivation and state the moral of this story.

Historical Note. Who was the first algebracist? Some say Diophantus is "The Father of Algebra". Diophantus lived in Alexandria, Egypt sometime in the early centuries AD. Others give the honor to al-Khwarzimi from Baghdad, who lived from around 780 to around 830 AD. Search the web for information on Diophantus and al-Khwarzimi.

CHAPTER 3

Quadratic Equations

1. Squares and Square Roots

For any positive number x, $x^2 = x \times x$ is a positive number. If x is negative then $x^2 = x \times x = -x \times -x$ is also positive. For example $(\frac{1}{2})^2 = \frac{1}{4}$ and $(-3)^2 = 9$. We note that two different numbers, any positive number and its negative, have the same square; for example, $(-\frac{1}{2})^2 = \frac{1}{4} = (\frac{1}{2})^2$. Hence if you know the square of a number, there are two possible values for the number itself; for example $x^2 = 4$ has two solutions, $x = 2$ and $x = -2$. Specifically, the equation $x^2 = a$ has two solutions when $a > 0$; it has one solution when $a = 0$ and it has no solutions when $a < 0$. We write $\sqrt{a}$ for the positive solution to $x^2 = a$ when $a > 0$, we have $\sqrt{0} = 0$ and, when $a < 0$, we say that $\sqrt{a}$ is an *imaginary* number or has no real number solution. In the case that a is positive, the negative solution to $x^2 = a$ is simply $-\sqrt{a}$ and we often write $\pm\sqrt{a}$ to indicate the set of both solutions to $x^2 = a$.

Exercise 3.1. Explain the following:

(i) $\sqrt{x}$ may not be defined, but when it is, we have $(\sqrt{x})^2 = x$;
(ii) $\sqrt{x^2}$ is always defined but may not equal x.

For a number x, $|x|$ (called the *absolute value* of x), is defined to be x when $x \geq 0$ and the positive number $-x$ when $x < 0$. Hence, $\sqrt{x^2} = |x|$.

The operation of taking the positive square-root of a positive number is different in several ways from the operations that we have studied so far. First of all when you add, subtract or multiply whole numbers you always get a whole number and when you add, subtract, multiply or divide fractions you always get another fraction. In contrast, the square-root of a positive whole number or fraction may be an entirely different kind of number: it will be a real number but it may not be a rational number. For example, $\sqrt{2}$ and $\sqrt{\frac{1}{2}}$ are not rational numbers. Real numbers that are not rational numbers are called *irrational numbers* and they have infinite, non-repeating decimal expansions.

To work with square-roots, we need to know the rules that they must follow.

Rule 10. If a is non negative and b is positive then:

(i) $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$;
(ii) $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.

Actually these rules follow from the first nine rules, but the derivation is straight forward and will be left as an exercise. It is not only important to know the rules that hold for square-roots; we need to know that some "rules" that you might think should hold really don't. If a and b are positive numbers then:

(i) $\sqrt{a} + \sqrt{b} \neq \sqrt{a + b}$;

(ii) $\sqrt{a} - \sqrt{b} \neq \sqrt{a-b}$.

Suppose that $\sqrt{a} + \sqrt{b}$ and $\sqrt{a+b}$ were equal; then their squares would be equal. We have:

$$\begin{aligned}(\sqrt{a}+\sqrt{b})^2 &= (\sqrt{a}+\sqrt{b}) \times (\sqrt{a}+\sqrt{b}) \\ &= \sqrt{a}\times\sqrt{a}+\sqrt{a}\times\sqrt{b}+\sqrt{b}\times\sqrt{a}+\sqrt{b}\times\sqrt{b} \\ &= a + 2\sqrt{ab} + b\end{aligned}$$

This will equal $a+b$, the square of $\sqrt{a+b}$, only if $2\sqrt{ab} = 0$, that is only if at least one of a or b were 0.

Exercise 3.2. Prove Rule 10 (i)

Exercise 3.3. Prove Rule 10 (ii)

Exercise 3.4. Prove that $\sqrt{a} - \sqrt{b} \neq \sqrt{a-b}$.

Working with square-roots may seem complicated at first. But after learning a few very simple tricks it becomes relatively east - maybe even fun, Think of the set of all rational numbers. They are *closed* under addition and multiplication. That is, if we carry out normal arithmetic operations (excluding $\sqrt{\ }$) with some rational numbers, the end result is always another rational number. We can build another closed number system by adjoining a specific square-root, say $\sqrt{5}$. So we look at all numbers of the form $a + b\sqrt{5}$ where a and b are rational numbers (e.g. $1 - \sqrt{5}$ or $\frac{4}{7} + \frac{3\sqrt{5}}{4}$). We wish to show that this set of numbers is closed under all of our basic operations. Clearly, it is closed under addition:

$$(a+b\sqrt{5}) + (c+d\sqrt{5}) = (a+c) + (b+d)\sqrt{5}).$$

Closure under multiplication is a bit more complicated:

$$(a+b\sqrt{5}) \times (c+d\sqrt{5}) = ac + ad\sqrt{5} + bc\sqrt{5} + bd\sqrt{5}^2 = (ac+5bd) + (ad+bc)\sqrt{5}.$$

The other requirement of a closed system is the existence of inverses. Additive inverses are obvious: the additive inverse of $a+b\sqrt{5}$ is $-a + (-b)\sqrt{5}$. But how do we see that, when $a + b\sqrt{5} \neq 0$, $\frac{1}{a+b\sqrt{5}}$ is actually a number of the form $c + d\sqrt{5}$?

$$\frac{1}{a+b\sqrt{5}} = \frac{a-b\sqrt{5}}{(a+b\sqrt{5})(a-b\sqrt{5})} = \frac{a-b\sqrt{5}}{(a^2-5b^2)} = \frac{a}{(a^2-5b^2)} - \frac{b}{(a^2-5b^2)}\sqrt{5}.$$

In other words:

$$(a+b\sqrt{5}) \times (\frac{a}{(a^2-5b^2)} - \frac{b}{(a^2-5b^2)}\sqrt{5}) = 1.$$

There is one fine point to this proof: we multiplied and divided by $a - b\sqrt{5}$ We took care to be sure that $a + b\sqrt{5} \neq 0$, but how can we be sure that $a - b\sqrt{5} \neq 0$? Well suppose that $a - b\sqrt{5} = 0$ or $a = b\sqrt{5}$. The left-hand side is rational; the right-hand side is irrational unless $b = 0$. In that case $a = 0$ too and $a + b\sqrt{5} = 0$. So, if $a + b\sqrt{5}$ is not equal to 0 then so is $a - b\sqrt{5}$.

Exercise 3.5. Verify that $\frac{a}{(a^2-5b^2)} - \frac{b}{(a^2-5b^2)}\sqrt{5}$ is the multiplicative inverse of $a + b\sqrt{5}$ by multiplying out the left hand side of the last equation.

Exercise 3.6. Simplify each of the following expressions, that is write them in the form $a + b\sqrt{5}$ where a and b are rational.

(i) $4 \times (1+\sqrt{5}) + (3 - 2\sqrt{5})$

(ii) $(1+\sqrt{5}) \times (3-2\sqrt{5})$
(iii) $(3-2\sqrt{5}) \div (1+\sqrt{5})$
(iv) $(1+\sqrt{5})^2$

As we noted at the start of this section, the rational numbers are not closed under square-roots, that the square-root of a rational number may not be another rational number. But sometimes it is and then we say that the number is a *perfect square* $\frac{1}{4}$ and 16 are perfect squares. The same is true of our new number system: most numbers of the form $a + b\sqrt{5}$ are not perfect squares but some are. For example, $6 - 2\sqrt{5}$ is a perfect square. To check this we write $6 - 2\sqrt{5} = (x + y\sqrt{5})^2$ and try to figure out how to choose x and y so that this equality holds. Expanding $(x + y\sqrt{5})^2$ we get $x^2 + 5y^2 + 2xy\sqrt{5}$ and we see that we must choose x and y so that $x^2 + 5y^2 = 6$ while $2xy = -2$. There are two obvious choices $x = 1$, $y = -1$ and $x = -1$, $y = 1$ giving $1 - \sqrt{5}$ and its negative $-1 + \sqrt{5}$ as the two square-roots of $6 - 2\sqrt{5}$.

EXERCISE 3.7. Compute $\sqrt{1 + \frac{4\sqrt{5}}{9}}$

2. The Solution Set of a Quadratic Equation

In the last chapter, we came across the quadratic equation $x^2 - \dfrac{P}{2}x + A = 0$, where P and A were the perimeter and area of a rectangle. The two solutions to this quadratic equation were then the length and width of that rectangle. To see that this is true, we observe that

$$(x-\ell)(x-w) = x^2 - x\ell - xw + \ell w = x^2 - (\ell + w)x + A = x^2 - \frac{P}{2}x + A$$

and note that $x^2 - \dfrac{P}{2}x + A$ equals zero if and only if one of $(x - \ell)$ or $(x - w)$ equals zero, that is, if and only if $x = \ell$ or $x = w$.

To be very precise, we define a quadratic equation in the variable x to be an equation of the form $ax^2 + bxy + c = 0$, where a b and c are constants and $a \neq 0$. A solution to $ax^2 + bx + c = 0$ is called a *zero* of the quadratic polynomial $ax^2 + bx + c$ In general, the quadratic equation $ax^2 + bx + c = 0$ has a solution if and only if the expression $ax^2 + bx + c$ can be factored. Before we show this, we note that by dividing through by a, yields the simpler quadratic equation $x^2 + Bx + C = 0$, where $B = \dfrac{b}{a}$ and $C = \dfrac{c}{a}$. Furthermore, the two quadratic equations $ax^2 + bx + c = 0$ and $x^2 + Bx + C = 0$ have exactly the same solutions.

First assume that $x^2 + Bx + C$ can be factored, that is, $x^2 + Bx + C = (x - r)(x - s)$ for the numbers r and s. Then by substitution we see that if $x = r$, i.e. if $x - r = 0$, we have $x^2 + Bx + C = (x - r)(x - s) = 0$. So r is a solution to $x^2 + Bx + C = 0$. By a similar argument s is also a solution. Now suppose that t is any other number. Then $t^2 + Bt + C = (t - r)(t - s)$ and since both $(t - r)$ and $(t - s)$ are different from 0, $(t - r)(t - s)$ is different from 0 and t is not a solution!

Next observe that, if r is a solution to $x^2 + Bx + C = 0$ then $-B - r$ is also a solution. To show this we must show that $(-B - r)^2 + B(-B - r) + C = 0$. Expanding and simplifying we have

$$\begin{aligned}(-B-r)^2 + B(-B-r) + C &= B^2 + 2Br + r^2 + (-B^2 - Br) + C \\ &= r^2 + Br + C \\ &= 0\end{aligned}$$

Finally, we observe that, if r is a solution to $x^2+Bx+C=0$ then $x^2+Bx+C=(x-r)(x-s)$ where $s=-B-r$. Before we multiply this out we note that since r is a solution $r^2+Br+C=0$ and $C=-r^2-Br$.

$$\begin{aligned}(x-r)(x-s) &= (x-r)(x+B+r)\\ &= x^2+xB+xr-rx-rB-r^2\\ &= x^2+Bx+(-r^2-Br)\\ &= x^2+Bx+C\end{aligned}$$

To summarize we have shown:

(i) If $x^2+Bx+C=(x-r)(x-s)$ then r and s are the zeros of x^2+Bx+C and the solutions to $x^2+Bx+C=0$ and they are the only zeros/solutions.
(ii) If r is a zero of x^2+Bx+C then $s=-B-r$ is also a zero of x^2+Bx+C.
(iii) If r is a zero of x^2+Bx+C then $x^2+Bx+C=(x-r)(x-s)$ where $s=-(B+r)$.
(iv) $x^2+Bx+C=0$ has at most two solutions.

Exercise 3.8. In each case, you are given one zero of a quadratic expression. First, verify that it is a zero by substitution then find the other zero and finally write out the factorization of the quadratic expression.

(i) x^2+2x-3, $r=1$;
(ii) $x^2+\frac{1}{2}x-3$, $r=-2$;
(iii) x^2-2x-3, $r=-1$;
(iv) $2x^2+4x-6$, $r=-3$;
(v) $2x^2+x-6$, $r=\frac{3}{2}$;

3. Finding the Solution Set of a Quadratic Equation

We now understand the nature of the solutions to a quadratic equation. But we still don't know how to find those solutions! Before we can describe the general method for solving a quadratic, we must consider two special cases.

Suppose that $s=r$ above. Then $(x-r)^2=x^2-2rx+r^2$. We say that r is a *double zero* and that the quadratic expression is a *perfect square.* We can easily recognize when a quadratic has a double zero:

x^2+Bx+C has a double zero if and only if $C=\frac{B^2}{4}$ and $r=-\frac{B}{2}$ is that double zero.

Next suppose that $s=-r$. Then $(x-r)(x-(-r))=(x-r)(x+r)=x^2-r^2$;.This special case is even easier to recognize since $B=0$ and $C\le 0$

The zeros of x^2+Bx+C are negatives of one another if and only if $B=0$ and $C\le 0$, furthermore $r=\pm\sqrt{-C}$ are those zeros.

Exercise 3.9. Solve each of the following quadratic equations or explain why it has no solution.

(i) $x^2+4x+4=0$;
(ii) $x^2-7x+\frac{49}{4}=0$;
(iii) $x^2-4=0$;
(iv) $x^2+4=0$;
(v) $x^2=4$;
(vi) $x^2=-4$;
(vii) $4x^2-49=0$;

The fourth and sixth equations in the last exercise have no solution. We show that as follows. Suppose that r is a solution. Then $r^2+4=0$ or $r^2=-4$. But, for every number r, positive or negative, $r^2 \geq 0$. So neither $x^2=-4$ nor $x^2+4=0$ has a solution.

EXERCISE 3.10. In each case add a constant to both sides of the equation so that the quadratic expression on the left becomes a perfect square.

(i) $x^2+6x=0$;
(ii) $x^2+6x+3=0$;
(iii) $x^2+6x-3=0$;
(iv) $x^2+6x+13=0$;
(v) $4x^2-48x=0$;

The above technique is called *completing the square.*

We can now describe a method for solving any quadratic equation and we illustrate this method with $x^2-7x-5=0$

Step 1. Move the constant term to the right hand side of the equation: $x^2-7x=5$.

Step 2. Add to both sides the constant that completes the square on the left side: $x^2-7x+\frac{49}{4}=5+\frac{49}{4}$.

Step 3. write the left side as a perfect square and simplify the right side: $(x-\frac{7}{2})^2=\frac{69}{4}$.

Step 4. If the right side is negative, there is no solution; if the right side is 0 or positive, take the square root of both sides: $(x-\frac{7}{2})=\pm\frac{\sqrt{69}}{2}$.

We can report the answer in several ways:
$x=\frac{7}{2}\pm\frac{\sqrt{69}}{2}$;
$x=\frac{7}{2}-\frac{\sqrt{69}}{2}$ or $x=\frac{7}{2}+\frac{\sqrt{69}}{2}$;
the zeros of x^2-7x-5 are $\frac{7}{2}-\frac{\sqrt{69}}{2}$ and $\frac{7}{2}+\frac{\sqrt{69}}{2}$.

4. The Quadratic Formula

One of the important achievements of algebra is the development of a simple formula for the zeros of any quadratic expression. The zeros of the general quadratic expression ax^2+bx+c are given by the *quadratic formula:*

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

So, for example, take $3x^2+4x-4$. By the formula we have

$$x=\frac{-4\pm\sqrt{16+48}}{6}=\frac{-4\pm 8}{6}$$

or $\frac{-12}{6}=-2$ is one zero and $\frac{4}{6}=\frac{2}{3}$ is the other. They are easily seen to check:
$3(-2)^2+4(-2)-4=12-8-4=0$ and
$3(\frac{2}{3})^2+4(\frac{2}{3})-4=\frac{4}{3}+\frac{8}{3}-4=0$.

To derive the quadratic formula we simply solve $ax^2+bx+c=0$ by the method of completing the square:

$$\begin{aligned}
ax^2+bx+c &= 0\\
x^2+\tfrac{b}{a}x+\tfrac{c}{a} &= 0\\
x^2+\tfrac{b}{a}x+ &= -\tfrac{c}{a}\\
x^2+\tfrac{b}{a}x+(\tfrac{b}{2a})^2 &= (\tfrac{b}{2a})^2-\tfrac{c}{a}\\
(x+\tfrac{b}{2a})^2 &= \tfrac{b^2-4ac}{4a^2}\\
x+\tfrac{b}{2a} &= \pm\tfrac{\sqrt{b^2-4ac}}{2a}\\
x &= \tfrac{-b\pm\sqrt{b^2-4ac}}{2a}
\end{aligned}$$

The expression b^2-4ac is called the *discriminant* of the quadratic expression. If it is negative, its square root is imaginary and there are no real number solutions; if it is 0, there is just one double root; if it is positive, there are two distinct solutions.

EXERCISE 3.11. Solve each of the following quadratic equations

(i) $x^2-4x-21=0$;
(ii) $x^2-11x-21=0$;
(iii) $6x^2-11x-7=0$;
(iv) $6x^2-11x+6=0$;
(v) $6x^2-11x+5=0$

At this point we can return to our area-perimeter problem from the last chapter. The problem was to find the length and width of the rectangle with perimeter P and area A or show that no such rectangle exists. We showed that the length and width were the solutions to the quadratic equation $x^2-\dfrac{P}{2}x+A=0$. The discriminant or this quadratic is $\frac{P^2}{4}-4A$. We conclude that there is not rectangle with perimeter P and area A whenever $\frac{P^2}{4}-4A<0$, that is whenever $P^2<16A$. When $P^2=16A$, such a rectangle exists and it is a square with sides of length $\frac{P}{4}$. Finally, when $P^2\geq 16A$ there is such a rectangle and its dimensions are

$$\ell=\frac{P+\sqrt{P^2-16A}}{4} \quad\text{and}\quad w=\frac{P-\sqrt{P^2-16A}}{4}.$$

For example the 3×5 rectangle has perimeter $P=16$ and area $A=15$ and from these two parameters we may recover its dimensions:

$$\ell=\frac{16+\sqrt{16^2-16\times 15}}{4}=\frac{16+4}{4}=5 \text{ and } w=\frac{16-\sqrt{16^2-16\times 15}}{4}=\frac{16-4}{4}=3.$$

EXERCISE 3.12. Complete the following table of rectangle dimensions:

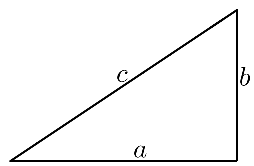
P	A	ℓ	w
25	35		
24	36		
25	40		
25	6		
25	1.24		

5. The Formula of Pythagoras

THEOREM 1. *The length of the hypotenuse of a right triangle with legs of length a and b is given by the formula*

$$a^2 + b^2 = c^2$$

.



PROOF. To begin, consider Figure 3.1. We know that the area of the outer square is $A_O = (a+b)^2 = a^2 + 2ab + b^2$. We also know that the area of the inner square is $A_I = c^2$ and that the area of each triangle is $A_T = \frac{1}{2}ab$. So we have

$$\begin{aligned} A_O &= A_I + 4A_T \\ a^2 + 2ab + b^2 &= c^2 + 4\left(\frac{1}{2}ab\right) \\ a^2 + 2ab + b^2 &= c^2 + 2ab. \end{aligned}$$

Thus subtracting $2ab$ from both sides, we have $a^2 + b^2 = c^2$. □

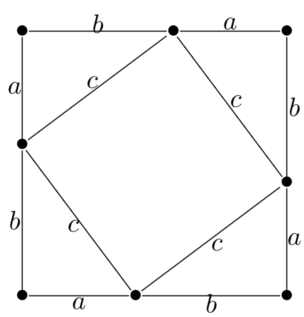


FIGURE 3.1. A Proof of the Pythagorean Theorem

EXERCISE 3.13. As above, a and b denote the lengths of the legs of a right triangle and c denotes the length of the hypotenuse.

(i) Find c if $a = 5$ and $b = 12$.
(ii) Find a if $b = 4$ and $c = 5$.
(iii) Find all possible right triangles with legs of equal length and 2 units less than the hypotenuse.
(iv) Find all possible right triangles with $c = \sqrt{7}$ and having one leg one unit longer than the other.

EXERCISE 3.14. [1] This exercise gives a geometric construction of the square root of a number given as a length. Consider Figure 3.2 below. Let x be the positive

[1] Adopted from (Ian Stewart, *Faggot's fretful fiaso*, in Music and Mathematics: From Pythagoras to Fractals, Edited by John Fauvel, Raymond Flood, and Robin Wilson, Oxford University Press, 2003)

number for which you wish to compute the square-root. Draw the segment AC with interior point D so that $|AD|$, the length of the segment AD is x and the $|DC|$ is 1. Construct the semicircle with AC as diameter and erect the perpendicular to AC at D. Let B denote the intersection of this perpendicular and the semicircle. Use the fact that the triangle ABC is a right triangle and express $|BD|$, in terms of x. Specifically, carry out the following steps:

(i) write $|AB|^2$ in terms of x and $|BD|$;
(ii) write $|BC|^2$ in terms of $|BD|$;
(iii) apply the formula of Pythagoras to triangle ABC and simplify.

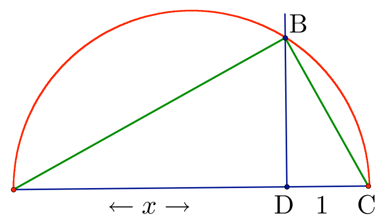
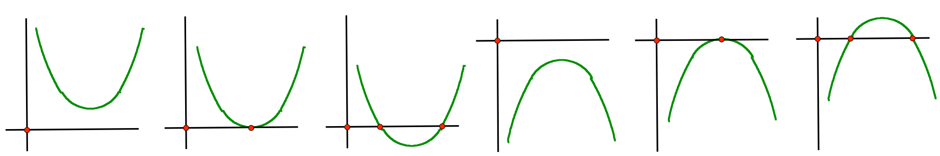


FIGURE 3.2. Square Root

6. Picturing Quadratic Functions

So far we have been considering quadratic equations (e.g. $x^2 - 6x + 8 = 0$) and computing their solutions ($x = 2$ and $x = 4$, in this case). But, it is natural to think of a quadratic expression as a polynomial function: $p(x) = x^2 - 6x + 8$ The values of x that give $p(x) = 0$ are called the *zeros* of the polynomial $p(x)$.

The graph of the quadratic function $f(x) = ax^2 + bx + c$ is a parabola. It opens upward if $a > 0$ and opens downward when $a < 0$. The zeros of this polynomial identify the points where the curve crosses the x axis. There are three possibilities for the zeros: no zeros, one double zero or two distinct zeros. This leads to the six possibilities pictured below:



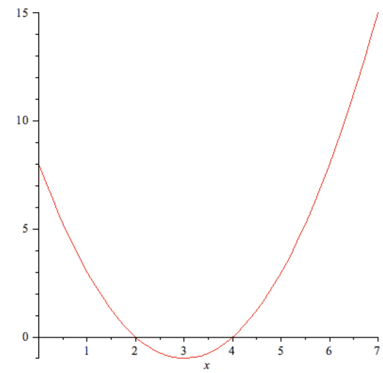
$a > 0$, no zeros	$a > 0$, double zero	$a > 0$, two zeros	$a < 0$, no zeros	$a < 0$, double zero	$a < 0$, two zeros

In the next section, we will want to know just when a quadratic function is positive, negative or zero. Once we picture the graph, such questions are easy to answer:

- If $a > 0$ and $f(x)$ has no zeros, $f(x)$ is always positive.
- If $a > 0$ and $f(x)$ has a double zero, $f(x)$ is always positive except at the zero.
- If $a > 0$ and $f(x)$ has two zeros $r < s$, $f(x)$ is always positive for $x < r$ and $x > s$, zero at r and s, and negative for $r < x < s$.
- If $a < 0$ and $f(x)$ has no zeros, $f(x)$ is always negative.

- If $a < 0$ and $f(x)$ has a double zero, $f(x)$ is always negative except at the zero.
- If $a < 0$ and $f(x)$ has two zeros $r < s$, $f(x)$ is always negative for $x < r$ and $x > s$, zero at r and s, and positive for $r < x < s$.

As an example consider $f(x) = x^2 - 6x + 8$ which has zeros 2 and 4. We conclude that $f(x)$ is positive for $x < 2$ and $x > 4$ and negative for x between 2 and 4:



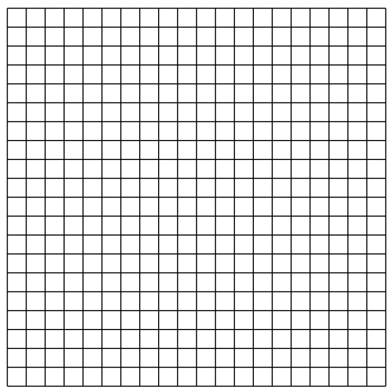
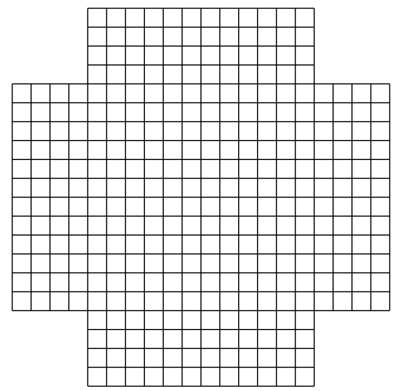
EXERCISE 3.15. For each of the following quadratic functions, decide when it takes on positive values and when it is negative. Then check your answers by graphing the quadratic.

(i) $f(x) = 4x^2 - 12x + 9$
(ii) $f(x) = -3x^2 + 10x - 9$
(iii) $f(x) = 2x^2 + 9x + 11$
(iv) $f(x) = -x^2 + 6x - 9$
(v) $f(x) = 2x^2 + 9x - 5$
(vi) $f(x) = -3x^2 + 10x - 3$

7. Calculus without the Calculus

Consider the following problem: You are given a 20×20 grid; from each corner you remove an $m \times m$ grid (Here m is restricted to be a whole number.); folding up the sides, you form an open topped box or tray. How should you choose m so that the box will have the largest volume?

In the example pictured below, $m = 4$. The volume of the box will then be $4 \times 12 \times 12 = 576$ cubic units. In general $V(m) = m(20 - 2m)^2$. In the case where m can be any real number, finding a value for m that gives the largest volume is a typical calculus problem. However, it can be solved using only algebra. The business community has long used the concepts of *marginal* cost and *marginal* profit, the additional cost or profit from producing one more widget (or whatever it is that they produce). We can easily adopt the technique of “marginals” to our problem.

Suppose we consider the box for a given m and ask the question: Should we increase the value of m by 1? In other words: is the volume of the box made with the $(m+1)\times(m+1)$ corners cut out larger than the volume of the box with the $m \times m$ corners removed? To answer this question, we compute the difference in volume:

$$\begin{aligned} V(m+1) - V(m) &= (m+1)(20-2m-2)^2 - m(20-2m)^2 \\ &= 12m^2 - 148m + 324. \end{aligned}$$

So if for a given value of m $12m^2 - 148m + 324$ is positive, we should increase the value of m by 1, otherwise we should not and might even consider decreasing the value of m by 1. To decide when $12m^2 - 148m + 324$ is positive and negative, we solve for its zeros using the quadratic formula:

$$\frac{148 - \sqrt{148^2 - 4\times 12\times 324}}{2\times 12} \quad \textit{and} \quad \frac{148 + \sqrt{148^2 - 4\times 12\times 324}}{2\times 12}$$

The resulting zeros are approximately 2.8 and 9.5. Since the coefficient of m^2 is positive, $12m^2 - 148m + 324$ will be positive for m equal 1 and 2 and for $m > 9$; it will be negative for $m = 3, 4, \ldots, 9$ So starting with 1, as m is increased by 1, the volume will increase until we reach 3 after that it will decrease until we reach 10. We have computed these volumes in the following table:

1	2	3	4	5	6	7	8	9	10
324	512	588	576	500	384	252	128	36	0

We could ask the same question for the 25×25 grid or the 30×30 grid; indeed for the $n\times n$ grid for any n. Algebra enables us to solve this problem for all such grids at the same time. The marginal change in volume for the $n \times n$ grid is given by:

$$(m+1)(n-2m-2)^2 - m(n-2m)^2 = 12m^2 - (8n-12)m + (n-2)^2.$$

Applying the quadratic formula gives the zeros:

$$\frac{8n - 12 - \sqrt{16n^2 - 48}}{24} \quad \textit{or} \quad \frac{8n - 12 + \sqrt{16n^2 - 48}}{24}$$

Which simplify to $\frac{2n-3-\sqrt{n^2-3}}{6}$ and $\frac{2n-3+\sqrt{n^2-3}}{6}$. We can fill in any value for n that we wish. When n is reasonably large, we can estimate these zeros by ignoring the -3 under the radical sign to get $\frac{n}{6} - \frac{1}{2}$ and $\frac{n}{2} - \frac{1}{2}$. So the volume should increase as

m increases until m reaches about $\frac{n}{6}$ then the volume will decrease until m reaches its largest possible value, $\frac{n}{2}$.

n	20	24	36	100	200
$\frac{n}{6}$	3.3	4	6	16.7	33.3
$\lceil\frac{2n-3-\sqrt{n^2-3}}{6}\rceil$	3	4	6	17	33

The expression $\lceil\frac{2n-3-\sqrt{n^2-3}}{6}\rceil$ stands for the smallest integer greater than or equal to $\frac{2n-3-\sqrt{n^2-3}}{6}$. In general, the *ceiling function* $\lceil\ \rceil$ rounds up to the next integer and the *floor function* $\lfloor\ \rfloor$ rounds down to the next integer.

EXERCISE 3.16. Solve the following problem: You are given a 20×30 grid; from each corner you remove an $m \times m$ grid; folding up the sides you form an open topped box. How should you choose m so that the box will have the largest volume?

EXERCISE 3.17. Solve the following general problem: You are given an $20 \times n$ grid; from each corner you remove an $m \times m$ grid; folding up the sides you form an open topped box. How should you choose m so that the box will have the largest volume?

EXERCISE 3.18. Solve the following general problem: You are given an $n \times 2n$ grid; from each corner you remove an $m \times m$ grid; folding up the sides you form an open topped box. How should you choose m so that the box will have the largest volume?

Now let's generalize the problem even further: You are given a 20×20 square; from each corner you remove an $x \times x$ square; folding up the sides you form an open topped box. How should you choose x so that the box will have the largest volume? Here we think of x as a continuous variable able to take on any real value between 0 and 10. As before we will compute the marginal increase in volume. But instead of requiring that we increase the value of x by 1, we will increase it by some small but unspecified amount ϵ. The difference in volume is now:

$$\begin{aligned} V(x+\epsilon) - V(x) &= (x+\epsilon)(20-2x-2\epsilon)^2 - x(20-2x)^2 \\ &= 12\epsilon x^2 + (12\epsilon^2 - 160\epsilon)x + 4\epsilon(\epsilon-10)^2. \end{aligned}$$

EXERCISE 3.19. Evaluate this last expression at $\epsilon = 1$ and compare it with our first computation of marginal change.

We interpret this two variable quadratic function

$$f(x,\epsilon) = 12\epsilon x^2 + (12\epsilon^2 - 160\epsilon)x + 4\epsilon(\epsilon-10)^2$$

as follows: if $f(x,\epsilon) > 0$, then the volume of the box obtained by cutting $(x+\epsilon)$ by $(x+\epsilon)$ squares from the corners is larger than the volume of the box obtained by cutting x by x squares from the corners. Now we find the zeros of this polynomial treating ϵ as a constant:

$$\frac{160\epsilon - 12\epsilon^2 \pm \sqrt{(160\epsilon - 12\epsilon^2)^2 - 4(12\epsilon)(4\epsilon(\epsilon-10)^2)}}{24\epsilon}.$$

this simplifies to

$$\frac{(4\epsilon)(40 - 3\epsilon \pm \sqrt{400 - 3\epsilon^2})}{24\epsilon}.$$

EXERCISE 3.20. Check the simplification.

At this point we think about choosing the size of the step ϵ. First we note that if $\epsilon = 0$, then $f(x, 0) = 0$ and the last expression above is undefined since it involves division by zero. On the other hand, if $\epsilon > 0$, however small, we may simplify the zeros further to get $\frac{40-3\epsilon-\sqrt{400-3\epsilon^2}}{6}$ and $\frac{40-3\epsilon+\sqrt{400-3\epsilon^2}}{6}$. We now ask what happens to these zeros as ϵ gets closer and closer to zero. The first fraction approaches $\frac{10}{3}$ while the second approaches 10. We may conclude that whenever $x < \frac{10}{3}$ the volume may be increased by increasing x by a sufficiently small amount but that whenever $x > \frac{10}{3}$ any increase of x will decrease the volume. It follows that we will maximize the volume by choosing $x = \frac{10}{3}$.

Calculus bundles this entire development into one simple operation: taking the derivative of $x(20 - 2x)^2$ moves directly to the quadratic polynomial $12x^2 - 160x + 400$ which has $\frac{10}{3}$ and 10 as its zeros.

Exercise 3.21. You are given a $w \times w$ square; from each corner you remove an $x \times x$ square; folding up the sides you form an open topped box. How should you choose x so that the box will have the largest volume?

Exercise 3.22. You are given a $\ell \times w$ rectangle ($\ell > w$); from each corner you remove an $x \times x$ square; folding up the sides you form an open topped box. How should you choose x so that the box will have the largest volume?

CHAPTER 4

Understanding Estimation

1. Rounding

The following problem was on a NY State test.
It takes Maria about 7 minutes to play a song on the piano. About how long will it take her to play it 3 times?

a. 10 minutes
b. 20 minutes
c. 30 minutes

One teacher concluded that since the possible answers were multiples of ten, students were expected to round to the nearest tens. So rounding 7 to the nearest tens gives 10 and 3×10 gives an answer of 30! It is clear from this simple problem that using rounding in numerical operations can lead to very strange conclusions. To help understand what is actually going on, let's review in some detail the basic idea of rounding.

Suppose that our restaurant bill is \$41.43 and we want to leave a tip of 15%. Since $.15 \times 41.43 = 6.2145$, it's not possible to tip exactly 15% of that amount. We will need to either round down and leave a tip of \$6.21 or round up and leave \$6.22. This type of situation arises regularly when dealing with monetary quantities, so rounding is commonplace in everyday commerce. Whether the exact amount is rounded up or down depends on the situation. If your grocer offers a special price of \$1.00 for three boxes of macaroni and cheese, you know you're going pay \$0.34 if you buy just one box. If a bank pays interest on \$127.85 at 4% compounded monthly, the interest due at the end of the first month is theoretically $127.85 \times .04/12 = 0.4261666...$ and the bank will very likely pay only \$0.42.

The decimal representation of a number, say 964.198, is an abbreviation for the number expanded in terms of powers of 10. That is,

$$9 \times 10^2 + 6 \times 10^1 + 4 \times 10^0 + 1 \times 10^{-1} + 9 \times 10^{-2} + 8 \times 10^{-3}$$

This number has 6 decimal digits. The coefficient of 10^2 is called the hundred's digit, the coefficient of 10 is the ten's digit, of 10^0 the unit's digit, of 10^{-1} the one-tenth's digit, of 10^{-2} the one-hundredth's digit, etc. The goal in rounding is to replace a number having many decimal digits with one that is close to it and has fewer decimal digits. A standard rounding method that is used in scientific calculations, is to round to the nearest acceptable number. Using this type of rounding to the nearest one-hundredth, all the numbers x that satisfy $6.21 < x < 6.215$ are rounded down to 6.21 and those that satisfy $6.215 < x < 6.22$ are rounded up to 6.22. What should be done with 6.215? Since numbers of the form $6.21d$ are rounded down to 6.21 for $d \in \{0, 1, 2, 3, 4\}$ and are rounded up to 6.22 for $d \in \{6, 7, 8, 9\}$, for the sake of balance we will round up also in the case of $d = 5$. Unless stated otherwise, we will use this method when rounding. With this method, the exact tip would be rounded to \$6.21, because 6.21 is closer 6.2145 than is 6.22; the interest in the example above would be rounded to \$0.43 because 0.43 is closer to 0.4261666... than is 0.42. If we round 964.198 to; the nearest unit, the result is 964, the nearest one-tenth, the result is 964.2, the nearest one-hundredth, the result is 964.20. Notice that after the last round, we have written the zero at the end of the number to indicate that rounding was to the nearest one-hundredth. We explore this further in the discussion below of significant digits.

Exercise 4.1. You were asked to round 8.3346 and you complied with 8.335. But then you were told "No, I meant round to the nearest 100th." So you rounded 8.335 to 8.34. Explain what went wrong!

Exercise 4.2. You are told that 57300 is the result of rounding the integer x to the nearest 100.

(i) What is the largest integer that x could be?
(ii) What is the smallest integer that x could be?
(iii) How many different integers are possible for the exact value of x ?

Exercise 4.3. Because of the high price of copper, it costs almost 2 cents to make a penny. Suppose that pennies were eliminated and all prices had to be rounded to multiple of 5 cents. One assumes that all prices will be rounded up. But assume that rounding is done to the nearest number of cents ending in 5 or 0 - call this the fair rounding scheme.

(i) Describe the fair rounding scheme for rounding to the nearest nickel.
(ii) Describe the fair rounding scheme for rounding to the nearest quarter.

Exercise 4.4. Consider rounding an integer to the multiples of 10 as a function from the integers to the multiples of 10. This is a many to one function. How many integers round to a specific multiple of 10 if

(i) you always round up;
(ii) you always round to the nearest multiple of 10 and all numbers ending in 5 are rounded up;
(iii) you always round to the nearest multiple of 10 and all numbers ending in 5 are so that the 10s digit is even.
(iv) experimentally discover which rounding rule is used by the round command on your calculator.

There are many reasons why one might wish to round to something other than the nearest 10th, 100th, etc.; for example, to use a standard ruler we may wish to round measurements given as decimals to measurements to the nearest 4ths, 8ths or 16ths. The following function will convert your calculator's *round* into a function that rounds to the nearest $\frac{1}{4}$: $Y_1(X) = round(4 \times X, 0)/4$

Exercise 4.5. Check that this function actually works. Explain how it works. Build a function that rounds to the nearest 8th, 16th, 3rd.

2. Approximations and Errors

When a number in decimal form is rounded to fewer digits, the rounded value is an approximation to the original value. There are many situations where we need to approximate the exact value of some quantity with another quantity. The difference between these two quantities is the *error* in the approximation. That is,

$$\text{Error}=\text{Exact Value - Approximation}$$

If any two of the values in this equation are known, the third is determined. The error could be positive or negative depending whether the estimate is less than or larger than the exact value.

Often, it is only the magnitude or size of the error that is of primary interest. Thus, we use the *absolute error*, which is defined by the following equation.

$$\text{Absolute Error} = |\text{Exact Value - Approximation}|$$

The absolute error is thus simply the distance on the number line between the exact value and the approximate value. The smaller the absolute error, the better the approximation.

The error that occurs when approximating the exact tip by rounding it down is: $6.2145 - 6.21 = .0045$. When rounding up the error is: $6.2145 - 6.22 = -.0055$. The absolute errors in these cases are .0045 and .0055 respectively.

Besides monetary calculations, approximations are used when making physical measurements and when doing calculator or computer calculations. Because a computer or a calculator can work with only finitely many numbers, some sort of truncation of decimals is needed when doing arithmetic operations.

In the real world, the magnitude of the error of an approximation, i.e. the absolute error, that is tolerable depends on the situation. Clearly the tolerable error in measuring a dose of liquid medicine is different from the tolerable error in measuring the volume of water in Lake Ontario. The concepts of relative error and relative absolute error are useful in such situations. These quantities are defined as follows:

$$\text{Relative Error} = \frac{\text{Error}}{|\text{Exact Value}|} \qquad \text{Absolute Relative Error} = \frac{|\text{Error}|}{|\text{Exact Value}|}$$

Exercise 4.6. View each of the following numbers as an approximation to the exact quantity 876.5437. Using your calculator, compute in each case, the error, the absolute error, the relative error and the absolute relative error.

(i) 876.5
(ii) 876.54
(iii) 876.543

Exercise 4.7. Using a standard ruler you measure the length of a piece of paper at 9 and $\frac{7}{8}$ inches to the nearest $\frac{1}{8}$ inch. Assuming your measurement is accurate:

(i) what is the maximum length that the paper could have?
(ii) what is the minimum length that the paper could have?
(iii) what is the largest the absolute error could be?
(iv) what is the smallest the absolute error could be?
(v) what is the largest the absolute relative error could be?
(vi) what is the largest the absolute relative error could be if you accurately measure the width of the paper to be 4 and $\frac{3}{8}$ inches.

3. Scientific Notation

Scientists have developed a convenient notation for working with numbers with magnitudes that might be very large or very small. This variation of the usual decimal notation writes a decimal number in the form

$$a \times 10^b$$

where a is a decimal fraction with $1 \leq |a| < 10$ and b is an integer. The quantity a is called the coefficient and is either positive or negative depending on whether the number itself is positive or negative. The quantity b is called the exponent and can also be positive or negative. With a restricted in this way, the notation is called normalized scientific notation. For example, using scientific notation, the number 964.198 would be written as 9.64198×10^2.

Typically, calculators and computers use some form of scientific notation to display numbers whose magnitudes are very large or very small. However, instead of showing 10 with its exponent, the display might simply indicate the exponent with E. For example on the TI-84, if you change the mode from NORMAL to SCI and enter 964.198 the result is 9.64198E2. Some calculators also use a variation of this scientific notation, called engineering notation, in which the exponent b is always a multiple of three. In engineering notation, 964.198 is already in standard form. Engineering notation is designed so that numbers are always read in units, thousands, millions, billions, thousandths, millionths, billionths, etc. For example 12345.678 in engineering notation is 12.345678E3 and is read as "12 point 345678 thousands."

Exercise 4.8. Rewrite each of these numbers in scientific and in engineering notation.

(i) 987654321
(ii) .0001234567

Exercise 4.9. Exactly which numbers will appear the same in scientific and in engineering notation? Give a careful description of this set by describing it in terms of intervals.

Exercise 4.10. Carry out the following computations by hand and report your answer in both scientific and engineering notation.

(i) $(8 \times 10^{17})(2 \times 10^6)$

(ii) $\dfrac{(8 \times 10^{17})}{(2 \times 10^7)}$

(iii) $\dfrac{(2 \times 10^7)}{(8 \times 10^{17})}$

What numbers can calculators or computers work with? We know there are infinitely many real numbers, but any computer can work with only finitely many numbers. Think of these numbers as represented in scientific notation. Such a number will then be of the form

$$\pm d_0.d_{-1}d_{-2}\ldots d_{-k} \times 10^b, \quad \text{where } d_0 \in \{1, 2, \ldots, 9\} \text{ and } d_{-i} \in \{0, 1, \ldots, 9\}.$$

Since every machine is finite, there will be some limit on the number of digits, k, and another limit on the possible integer exponents, b. Such a number is called a floating point number or a machine number. For the purposes of illustration, suppose we have a small computer where $k = 3$ and $|b| \leq 3$. How many numbers does this computer have? There are two possibilities for the sign, nine choices for d_0, ten choices for each of d_{-1}, d_{-2}, d_{-3}, and seven possible choices for b. The computer will also have the number 0, so it has a total of $9 \times 10^3 \times 7 + 1 = 63001$numbers. The largest number in the computer is $9.999 \times 10^3 = 9999$. The next smaller number is 9.998×10^3, and the distance between these is $0.001 \times 10^3 = 1$. The smallest positive number it has is 1.000×10^{-3} and the next larger number is $1.001 \times 10^{-3} = 0.000001$. So numbers are bunched much more closely together near 0 than they are far to the right on the number line. When an arithmetic operation is performed with two of these computer numbers, the result will often not be one of the numbers in the computer. So, it is rounded to the nearest number in the computer.

The TI84 and most computers and calculators carry in memory more digits than they report. They do this to retain accuracy. They actually make computations with the full set of digits in memory and after computations are perform they round to the number of digits they report. The TI84 gives each of π and e to 9 decimal places (reporting 10 digits). The following experiments show that the calculator is actually keeping in its memory more than these 10 digits.

- Be sure that your calculator's mode is set at NORMAL and FLOAT.
- Enter π and then retype that number from the keyboard and store it in memory P. So the key π represents the actual number stored in the calculator while the key P represents the number that the calculator reports for π.
- Do the same for e storing the displayed number in memory E.

EXERCISE 4.11. Compute and compare:

(i) $\pi + e$ and $P + E$
(ii) $\pi - e$ and $P - E$
(iii) $\pi \times e$ and $P \times E$
(iv) $\pi \div e$ and $P \div E$
(v) 3π and $3P$

This last pair of computations tells us that the calculator rounded up when reporting the 10 digits of π. A simple trick will enable you to discover just how many digits of π that your calculator actually has in its memory: Enter $\pi - 3$ and return; enter $\pi - 3.1$ and return; and so on.

EXERCISE 4.12. How many digits of a number does the TI84 display? How many does it actually keep in memory? Find all of the digits that the TI84 actually computes for $\sqrt{2}$.

Here is a general observation: If x is a number in the interval $[a, b]$, then the distance from x to the midpoint of the interval $[a, b]$, $(a + b)/2$, is at most $(b - a)/2$. From another point of view, either a or b, which ever is nearest x, gives an approximation to x with absolute error at most $(b - a)/2$.

Consider a positive number x in decimal form. How far away from x is the nearest machine floating number of the form above? If

$$x = (d_0.d_{-1} \dots d_{-k}d_{-(k+1)} \dots) \times 10^b,$$

then the x falls in an interval between two floating point numbers with spacing 10^{b-k}. Thus, from our general observation, the nearest floating point number is no further from x than $\frac{1}{2} \times 10^{b-k}$. That is, the absolute error in approximating x by the nearest floating point number is at most $\frac{1}{2} \times 10^{b-k}$. Because $\frac{1}{|x|} \geq 10^{-b}$, the relative error in this approximation is at most $\frac{1}{2} \times 10^{-k}$.

4. Measurements and Significant Digits

The accuracy of a measurement of a physical quantity depends on the precision of the measuring instrument and the ability of the technician to accurately read the result. No measuring instrument is infinitely precise and two careful technicians might read the result as slightly different. So the recorded measurement is an approximation or estimate of the exact measurement. For example, when an ordinary ruler is used to measure the length of the page of a book the end of the page probably won't fall exactly on a ruler mark. The measurer must then estimate mark at the end of the page. If the page length falls between 20.3 cm and 20.4 cm, the next decimal might be estimated or the closer of these values might be used as the approximate length of the page. If the measurement is estimated as 20.31 or 2.031×10 in scientific notation, it is implied that the exact length falls between 2.0305×10 and 2.0315×10, so the absolute error would be at most .005. Notice the difference between reporting the result as 20.31 versus reporting as 20.310; the latter value implies an error of at most .0005. Scientist indicate the accuracy of a measurement by giving the result in scientific notation with rules for identifying the significance of the digits.

A digit is considered significant if it is meaningful in a measurement or is an estimated digit in the measurement. For example, if the length of the page is given as 2.031×10 cm, the digits 2, 0, 3 and 1 are all significant digits; the first three were measured and the last was estimated. When interpreting the number of significant digits in a measurement or when recording a measurement, the following rules apply:

- The digits 1, 2, ... 9 are always significant.
- Zeros between two other significant digits are significant.
- Zeros to the right of the decimal place and to the right of another significant digits are significant.
- Zeros used for spacing are not significant.

Thus, if a measurement is reported as .0020640 the measurement has five significant digits. In scientific notation it would be written as 2.0640×10^{-3}, with all digits in this representation significant. To compute the absolute error we interpret the terminal 0 as the result of rounding. Hence the exact value of the quantity being measured is in the interval $[.00206395, .00206405)$. The computation $\frac{(.00206405-.00206395)}{2} = 5\text{E-}8$ tells us that the implied absolute error in this measurement is 5×10^{-8}.

If we report the distance from the earth to the sun, as about 93 million miles accurate to the nearest million miles, it should not be written as 93,000,000 because that would imply an error of at most one-half mile! It should be given as 9.3×10^7

Exercise 4.13. A web research project.

(i) Find the latest data on the maximum and minimum distances between the earth and the sun.
(ii) What is the absolute error implied by these measurements?
(iii) What is the error in the 9.3×10^7 mile approximation? the absolute error? the relative error? the absolute relative error?
(iv) Is 9.3×10^7 the best single number estimate to the distance between the earth and the sun? Specifically could you safely include another digit or two? Explain.

(v) Find the latest data on the maximum and minimum distances between the earth and the moon.
(vi) What is the absolute error implied by these measurements?
(vii) What number would you report if you were asked for a single number representing the distance between the earth and the moon.

5. Operations with Approximate Quantities

In this section we address the issue of the propagation of errors when we perform arithmetic operations on approximate values rather than the corresponding exact numbers. That is , if x and y are the exact quantities and x^* and y^* respective approximations, how well does $x^* + y^*$ approximate $x + y$ and how well does $x^* \times y^*$ approximate $x \times y$. Common sense tells us that we can not expect a sum or product to be more accurate than the individual approximations.

First, suppose the numbers are approximations written in scientific notation so the significant digits can be identified. The rule for addition and subtraction of two such quantities, is to round to the least-accurate decimal digit, that is to the final decimal digit of the least accurate of the two quantities. For example $1.653 \times 10^{-1} + 7.19 \times 10$ would be added as follows:

$$\begin{array}{r} 0.1653 \\ +76.9 \\ \hline 77.0653 \end{array}$$

The least accurate of these two numbers is the 76.9, it is only accurate to the nearest 10th while 0.1653 is accurate to the nearest 10 thousandth. So the least-accurate digit is 9 and we report the answer to the nearest 10th. The sum is rounded to 77.1.

We can explain this convention by keeping track of the errors. Now suppose that the error bounds $|x - x^*| \le \varepsilon_x$ and $|y - y^*| \le \varepsilon_y$ are valid. Then,

$$|(x \pm y) - (x^* \pm y^*)| \le |x - x^*| + |y - y^*| \le \varepsilon_x + \varepsilon_y$$

That is, the error in the sum is bounded by the sum of the error bounds. Similarly, the error in the difference is bounded by the sum, *not the difference*, of the error bounds. Applying this analysis to the above example, let $x^* = 0.1653$ with an error bound $\varepsilon_x = 0.00005$ and let $y^* = 76.9$ with the error bound of $\varepsilon_y = 0.05$. The small error in the approximation of the summand $x^* = 0.1653$ is insignificant when compared to the relatively large error in the approximation of the summand $y^* = 76.9$

The rule for multiplication and division is that the product should be rounded to the least number of significant digits in each factor. Thus, the product of 2.812×10 and 1.7×10^{-2} would be rounded to 4.8×10^{-1}.

Numbers that are exact, such as integers that are determined by counting rather than measurements, are assumed to have infinitely many significant digits. Thus, if a rod has length 2.82 meters, then the length of 4 such rods is 11.28 rods.

Since,

$$x \times y - x^* \times y^* = x \times (y - y^*) + (x - x^*) \times y - (x - x^*) \times (y - y^*),$$

taking absolute values gives

$$\begin{aligned} |x \times y - x^* \times y^*| &\le |x| \times |y - y^*| + |x - x^*| \times |y| + |x - x^*| \times |y - y^*| \\ &\le |x| \times |y - y^*| + |y| \times |x - x^*| + \varepsilon_1 \times \varepsilon_2. \end{aligned}$$

Dividing by $|x \times y|$ and simplifying we obtain,

$$\frac{|x \times y - x^* \times y^*|}{|x \times y|} \le \frac{|y - y^*|}{|y|} + \frac{|x - x^*|}{|x|} + \frac{\varepsilon_1 \times \varepsilon_2}{|x \times y|}.$$

If the two error bounds are small relative to the exact values, which is typically the case, then their product is essentially negligible. In that case we have a rule of thumb that the relative error in the product is bounded by the sum of the relative errors. It is worth noting that if one of the numbers is exact, then the relative error in the product is the same as the relative error of the approximate factor.

Now, let's return the test problem that was described at the beginning of this chapter. If we interpret 7 as a measurement reported using significant digits, then we conclude that the actual playing time is between 6.5 and 7.5, The number 3 is exact, so the time for playing 3 times is between 19.5 and 22.5. Of the choices stated in the problem, 20 minutes is the only one in this range.

What can we say about the relative error in total playing time? Since we don't know the exact time for playing the piece once we can't compute exactly the relative error of the total playing time, but we can compute an upper bound for it. What is the largest possible relative error for playing the piece once? That is, what is the largest value the fraction

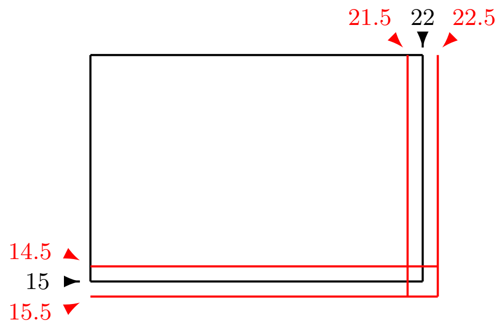
$$\frac{|\mathsf{Error}|}{|\mathsf{Exact\ Value}|}$$

could have? The fraction can be no larger then the largest possible value of $|\mathsf{Error}|$ divided by the smallest possible value of $|\mathsf{Exact\ Value}|$. We have already observed in this case that the largest $|\mathsf{Error}|$ can be is 0.5 while the smallest that $|\mathsf{Exact\ Value}|$ can be is 6.5. Therefore, the absolute relative error is at most $\frac{0.5}{6.5} = \frac{1}{13} \sim .077$. In reporting an error, one often uses percentages. So we might say that it takes about 7 minutes to play the piece and that our estimate is accurate to within 7.7%. Now, since 3 is exact, the relative error in the approximation for the total playing time is no more than $\frac{1}{13}$.

If the piece were longer, say about 20 minutes, the the absolute relative error would be smaller. Of course we could easily compute a bound on the absolute relative error in this case too. We could do the same for a piece running approximately 36 minutes and so on. Again using the power of algebra, we can cover all cases in just one computation! Suppose that the rounded time to play a piece is m minutes. Then the absolute relative error is bounded by the fraction $\frac{0.5}{m-0.5} = \frac{1}{2m-1}$. In our case $m = 7$, and we get the bound $\frac{1}{13}$; for a 20 minute piece we get $\frac{1}{39}$ or about 2.6%; a 50 minute piece has its absolute relative error bounded by $\frac{1}{99}$, about 1%, and so on.

Now let's consider another estimation problem. Suppose that you measure the length ℓ and width w of a rectangular garden plot and round off each measurement to the nearest foot. Using the rounded measurements, how far off could your computation of the perimeter and area be?

Let's look at the perimeter first and let's take a specific example. Say that your rounded measurements are 22 ft. long by 15 ft. wide. The estimated perimeter is $2 \times (22 + 15)$ or 74 ft. The error in length could be as much as 6 inches and the same for the width. So $\ell + w$ could be in error by as much as a foot. The number 2 is exact, so the error of perimeter is at most 2. Specifically, the dimensions could



be as small as 14.5 feet by 21.5 feet for a perimeter of 72 feet (see the inside red rectangle in the next figure) and the dimensions could be as large as 15.5 feet by 22.5 feet for a perimeter of 76 feet (see the outside red rectangle in the figure). Of course, the error in ℓ could partially cancel the error in w — if one measurement was too large and the other too small. On the other hand. as we just pointed out, the measurements could both be too large or both too small and then errors would accumulate.

Now consider the area. One might guess that you could be off by as much as a square foot. But, to understand exactly what could happen we must rely on an analysis of the error terms. Let e denote the error in measuring length, that is the actual length ℓ is given by $\ell = 22 + e$. Let f denote the error in measuring width, so $w = 15 + f$ Then

$$\ell \times w = 22 \times 15 + 22f + 15e + e \times f.$$

So the computed area, $22 \times 15 = 330$ sq. ft., will be off by $22f + 15e + e \times f$. As we mentioned before the term $e \times f$ will be significantly smaller than either of the other two terms and so we ignore it and concentrate on the remaining two terms. If both measurements were short by 6 inches this error term is 18.5 square feet. The inner rectangle has area $21.5 \times 14.5 = 311.75$ sq. ft. or 18.25 sq. ft. less than the estimation of 330 sq. ft. The outer rectangle has area $22.5 \times 15.5 = 348.75$ sq. ft. or 18.75 sq. ft. more than the estimation of 330 sq. ft.

Exercise 4.14. Explain why these two extreme cases are off by different amounts. Explain why they are both different from the 18.5 we computed.

6. Other Applications

A common situation in mathematics is that some number is theoretically known to exist and decimal or general rational approximations to it are sought. A solution of an equation is a case in point. If the number is known to be irrational, it is natural to look for a rational approximation.

For example, we know that $\sqrt{2}$ is irrational. Let's look for rational approximations to it. We know that $1 < \sqrt{2}$ because for any number $r \leq 1$ we have $r^2 \leq 1$. Similarly, $\sqrt{2} < 2$ because the square of any number $r \geq 2$ is at least 4. Thus, we could take $\frac{3}{2}$ as a first approximation to $\sqrt{2}$. Since $(\frac{3}{2})^2 = \frac{9}{4} > 2$, we conclude that $\sqrt{2} \in [1, \frac{3}{2}]$ and that the absolute error in the approximation of $\sqrt{2}$ by $\frac{3}{2}$ is at most $\frac{1}{4}$. An ad hoc method of getting a more precise bound of the error is to observe that

$$|A^2 - B^2| = |A - B||A + B| \quad \text{giving} \quad |\mathrm{A} - \mathrm{B}| = \frac{|\mathrm{A}^2 - \mathrm{B}^2|}{|\mathrm{A} + \mathrm{B}|}.$$

With $A = \sqrt{2}$ and $B = \frac{3}{2}$, this can be rewritten as

$$|\sqrt{2} - \frac{3}{2}| = \frac{2 - 2.25}{\sqrt{2} + 1.5} \le \frac{0.25}{2.5} = 0.1.$$

The right fraction was obtained by replacing $\sqrt{2}$ by 1 and observing that decreasing the denominator increases the value of the fraction, hence the inequality.

Exercise 4.15. Carry out the above error analysis for the approximation $\frac{7}{5}$.

One of the most estimated numbers of all time is π, the ratio of the circumference of a circle to its diameter. The Babylonians used $\frac{25}{8} = 3.125$. Archimedes of Syracuse bounded π between $\frac{223}{71}$ and $\frac{22}{7}$ and then used the average of these two numbers as an estimate for π. An early estimate from Asia (India or perhaps China) is $\frac{355}{113}$. Your calculator has a built in estimate for π accurate to whatever number of decimal places your calculator keeps.

Even though the number that your calculator holds for π is itself an estimate assume it to be the correct value for π in working this next exercise.

Exercise 4.16. Calculate the error, the absolute error, the relative error and the absolute relative error for each of the following approximations to π. Also state the accuracy of the estimate in terms of the number of correct digits in its decimal expansion.

(i) $\dfrac{25}{8}$;

(ii) $\dfrac{22}{7}$;

(iii) $\dfrac{223}{71}$;

(iv) the average of the previous two estimates;

(v) $\dfrac{355}{113}$.

Even though your calculator can hold in memory only so many digits of a number, you can use the calculator to make computations accurate to as many decimal places as you wish. The TI84 displays 10 digits and holds 14 in memory. We will use only the 10 displayed digits. Suppose that we wish to add two 15 digit numbers accurately. Say $m = 987,654,321,012,345$ and $n = 864,209,753,187,654$. Let $w = 987,654$ and $x = 321,012,345$, then $m = w10^9 + x$. Similarly $n = y10^9 + z$, where $y = 864,209$ and $z = 753,187,654$. Then $m + n = (w + y)10^9 + (x + z)$. Computing $(x + z)$ on the calculator gives 1, 074,199,999. Notice, we must carry a 1 into the 10^{10} column. So we calculate $w + y + 1$ to get 1,851,864. and we conclude that $m + n = 1,851,864,074,199,999$.

While the TI84 holds only 14 digits in memory, you can enter as large a number as you wish. So you can enter $987,654,321,012,345 + 864,209,753,187,654$. The calculator will round the answer to 1,851,864,074,200,000. and display 1.851864074E15. To check that indeed the last digit is 2, compute

$$987,654,321,012,345 + 864,209,753,187,654 - 1.8 \times 10^{15}$$

to get 5.18640742E13. One last observation: had we split the digits of our two large numbers into groups of 5 and 10 instead of 6 and 9, we would have lost a digit because there would be a carry over: $4,321,012,345 + 9,753,187,654 = 14,074,199,999$ an 11 digit number. Hence the calculator will round and report 1.40742E13. In general, the sum of two k digit numbers will be a k or $k+1$ digit number; so, in parsing the numbers, one must take into account the possible sizes of the intermediate answers.

EXERCISE 4.17. Compute all digits of the following sums:

(i) 987,654,321,987,654,321 + 776,655,443,322,110,099
(ii) 987,654,321,987,654,321,987 + 776,655,443,322,110,099,887
(iii) 987,654,321,987,654 + 321,987,776,655,443 + 322,110,099,887,766

The product of two 8 digit numbers will be a 15 or 16 digit number and hence out of range of the TI84. Again, by splitting the numbers we may compute all digits of the answer. Let $m = 87,654,321$ and $n = 53,187,654$. Write $m = 8765 \times 10^4 + 4321$ and $n = 5318 \times 10^4 + 7654$ or symbolically $m = w10^4 + x$ and $n = y10^4 + z$, Then $mn = (w10^4 + x)(y10^4 + z) = wy10^8 + (wz + xy)10^4 + xz$. Computing the products we have $wy = 46612270$, $wz + xy = 90066388$ and $xz = 33072934$. Now we must add:

```
4,661,227,000,000,000
      900,663,880,000
           33,072,934
```

We will have to break this addition up into pieces. Start by adding the last 9 digits to get $663,880,000 + 33,072,934 = 696,952,934$ and then the first 7 digits to get $4,661,227 + 900 = 4,662,127$. So our or product is 4,662,127,696,952,934.

EXERCISE 4.18. Compute all digits of the following products:

(i) $987,654,321 \times 123,456,789$
(ii) $987,654,321,987 \times 776,655,443,322$

CHAPTER 5

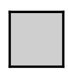
Discovering Formulas

In this section we indicate how several interesting formulas can be discovered. We suppose that we have a large supply, as many as needed, of square tiles. We describe several ways of laying them out in patterns. These patterns are developed in steps and at the end of each step the total number of tiles in the pattern is counted. The goal is to determine the number of tiles used after a fixed number of steps and to express that number as a formula in n, the number of steps.

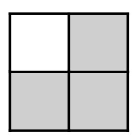
1. Formulas from areas of tile patterns

Consider the following sequence of tile patterns.

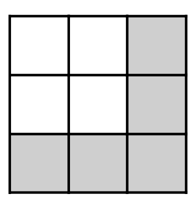
- Step 1: Place one tile:



- Step 2, Place three tiles around the right side and the bottom of the first tile:

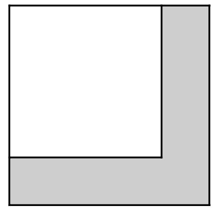


- Step 3, Place 5 tiles to the right side and the bottom of previously placed tiles:



⋮

- Step n: Place $2n-1$ tiles to the right side and the bottom of the previously placed tiles.

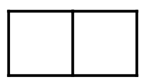


The table below shows the count and formula for the total number of tiles used after completing the steps listed.

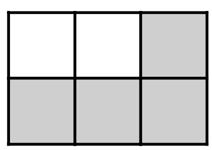
	Step 1	Step 2	Step 3	$\cdots$	Step n
Tiles	1	$1+3=4$	$1+3+5=9$	$\cdots$	$1+3+\cdots(2n-1)=n^2$

We can use the same idea to compute the formula for the sum of consecutive even numbers.

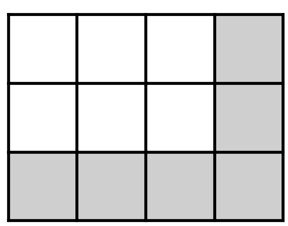
- Step 1, Place two tiles in a horizontal row:



- Step 2, Place one tile on the right side of the two tiles and then place three tiles on the bottom of the previously placed tiles:



- Step 3, Similarly, place 6 tiles to the right and bottom of the previously placed tiles:

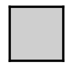


Exercise 5.1. Consider the above sequence of steps.

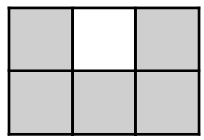
(i) Carryout the next two steps.
(ii) Make a table listing the step number and the total number of tiles placed after that step.
(iii) Find the formula for the number of tiles placed after the $n$the step.
(iv) Modify the formula you found to obtain a formula for the sum of the first n consecutive integers.

Here is another formula from a tile pattern.

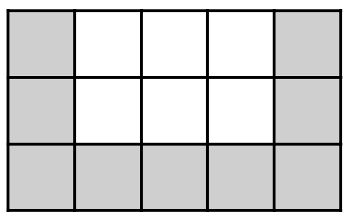
- Step 1, Place one tile:



- Step 2, Place 5 tiles to the right side, the left side and the bottom of the previously placed tiles:



- Step 3, Similarly, place 9 tiles to the right and bottom of the previously placed tiles:



Exercise 5.2. Consider the above sequence of steps.

(i) Carryout the next two steps.
(ii) Show that at each stage you are adding a number of the form $4n + 1$ - starting with $n = 0$.
(iii) Make a table listing the step number and the total number of tiles placed after that step.
(iv) Find the formula for the number of tiles placed after the $n$the step.

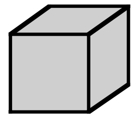
Exercise 5.3. For another formula-by-pattern, start by placing 2 tiles horizontally. In step 2, place 6 tiles on the right side, the left side and the bottom of the previously placed tiles. Continue the pattern for several steps. Make the table and find the formula for the total number of tiles.

For still another formula-by-pattern, start by placing a single tile in step 1 and then in subsequent steps surround the previously placed tiles on all sides. Continue the pattern for several steps. Make the table and find the formula for the total number of tiles.

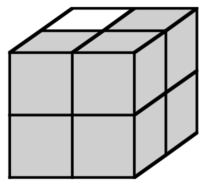
2. Formulas from areas of tile patterns

Now we look at a 3-dimensional extension of tile patterns. Instead of tiles we will be producing patterns of cubes.

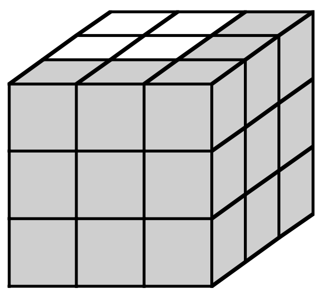
- Step 1, Place a cube in space:



- Step 2, Place a cube on each of two visible vertical faces, place another cube adjacent to these two, and place four cubes on the bottom to form a $2 \times 2 \times 2$ cube.



- Step 3, Place four cubes on each of the two visible vertical faces, two cubes between them, and nine cubes on the bottom to obtain a $3 \times 3 \times 3$ cube:



- Step n, Place $(n-1)^2$ cubes on each of two visible vertical faces, $(n-1)$ cubes between them, and n^2 on the bottom to obtain an $n \times n \times n$ cube. That is, we add
$$2(n-1)^2 + (n-1) + n^2 = 3n^2 - 3n + 1$$
cubes in step n. This means
$$\sum_{k=1}^{n} (3k^2 - 3k + 1) = n^3$$
- Use the previously obtained formula for the sum of consecutive integers to show that
$$\sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{6}$$

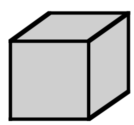
Consider the steps outlined above that resulted in an $n \times n \times n$ cube after n steps. Suppose that the individual cubes are held together to form a solid cube which is painted on all sides. Now we want a count, after each step, of the number of individual cubes that are painted on 0, 1, 2, ...5,or 6 sides. We make a table as before and include the total number of cubes counted and the total area which serves as a check on our calculations.

Step	0	1	2	3	4	5	6	No. Cubes	Area
1							1	1	6
2				8				8	24
3	1	6	12	8				27	54
4	8	24	24	8				64	96
n	$(n-2)^3$	$6(n-2)^2$	$12(n-2)$	8				n^3	$6n^2$

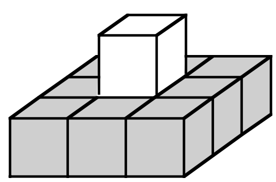
Notice that at step n the check is that the total number of sides painted equals the surface area of the cube. That is, $0*(n-2)^2+1*6(n-2)^2+2*12(n-2)+3*8 = 6n^2$

Next we consider a three dimensional analogue of the last tile pattern.

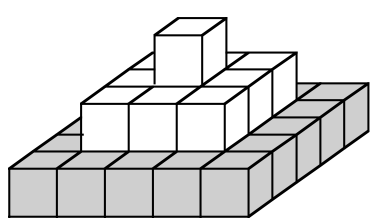
- Step 1, we place one cube.



- Step 2, place a 9-cube square underneath the cube placed in Step 1 and with that cube in the center.



- Step 3, place a 25-cube square underneath the previously placed cubes with them in the center.

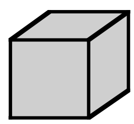


- Again, after each step, we want a count of the number of individual cubes that are painted on 0, 1, 2, ...5, or 6 sides. We make a table as before and include the total number of cubes counted and the total area.

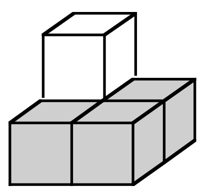
	Step 1	Step 2	Step 3	$\cdots$	Step n
0 Sides Painted			1		$\frac{(n-2)(2n-3)(2n-5)}{3}$
1 Side Painted		1	9		$(2n-3)^2$
2 Sides Painted			4		$4(n-2)^2$
3 Sides Painted		4	16		$12n-20$
4 Sides Painted		4	4		4
5 Sides Painted		1	1		1
6 Sides Painted	1				
Total Cubes	1	10	27		$\frac{4n^3-n}{3}$
Surface Area	6	34	86		$12n^2-8n+2$

Finally, another three dimensional analogue of one of the tile patterns.

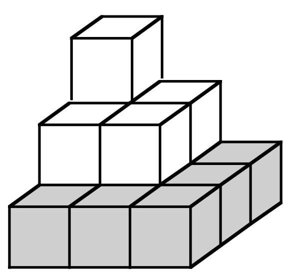
- Step 1, we place one cube:



- Step 2, place a 4-cube square underneath the cube placed in Step 1 and with that cube in a corner:



- Step 3, place a 9-cube square underneath the previously placed cubes at the same corner;



- Continue in this way.

	Step 1	Step 2	Step 3	$\cdots$	Step n
0 Sides Painted					$\frac{2n^3-15n^2+37n-30}{6}$
1 Side Painted			1		$2n^2-9n+10$
2 Sides Painted			3		n^2-2n
3 Sides Painted		1	6		$5n-9$
4 Sides Painted		3	3		3
5 Sides Painted		1	1		1
6 Sides Painted	1				
Total Cubes	1	5	14		$\frac{2n^3+3n^2+n}{6}$
Surface Area	6	20	42		$4n^2+2n$

3. The Area of a Region with Curved Boundary

In this section we show how the formulas we found can be used to determine the area of the region that lies between the parabola $y = x^2$ and x-axis for $0 \le x \le 1$. The basic idea for approximating such areas goes back more than 2,000 years to Archimedes of Syracuse! This is how he analyzed the problem. Divide the interval [0,1] into n subintervals of equal length, i.e. of length $\frac{1}{n}$. On the $i$the subinterval, construct a rectangle with base on the x-axis, and of width $\frac{1}{n}$ and height $\left(\frac{i}{n}\right)^2$. (See the figure.) The area of this rectangle is $\frac{i^2}{n^3}$.

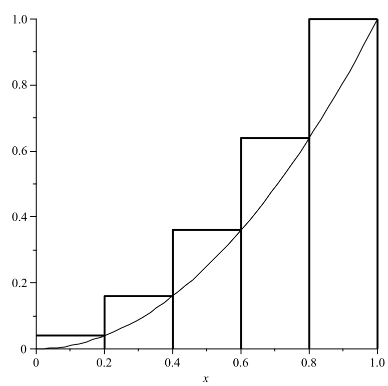


FIGURE 5.1. Outer Rectangular Approximation

Clearly the area under the parabola, call it A, is less than the sum of the areas of these n rectangles. That is, the quantity

$$\sum_{i=1}^{n} \frac{i^2}{n^3} = \frac{1}{n^3}\frac{2n^3+3n^2+n}{6} = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

is greater than A.

Now construct on the $i$the subinterval another rectangle of height $\frac{i-1}{n}$. The area of this rectangle is $\frac{(i-1)^2}{n^3}$ and the sum of the smaller rectangles is less than A. So, the quantity

$$\sum_{i=1}^{n} \frac{(i-1)^2}{n^3} = \frac{1}{n^3}\frac{2(n-1)^3+3(n-1)^2+(n-1)}{6} = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$$

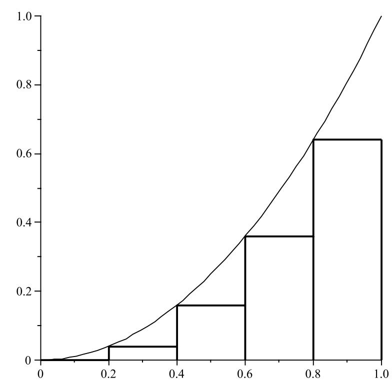


FIGURE 5.2. Inner Rectangular Approximation

is less than A. Now these results hold for any integer n. When n is very large, both of rectangular areas are near $\frac{1}{3}$ and the larger n becomes the closer these areas are to $\frac{1}{3}$. So we conclude that $A = \frac{1}{3}$.

4. The Trapezoid Rule

The method described above for finding the area bounded by the parabola and the x-axis gives the area as the limiting value of approximating areas. In many applications, it is not possible to obtain an exact value and approximations must be used. Schemes for finding good approximations are often complicated and computers are used to obtain them. Here we describe a variation of the method of Archimedes for calculating the area of a region with a curved boundary.

We suppose the region is bounded by the x-axis and the curve $y = f(x)$ for $a \leq x \leq b$. As before, we divide the interval $[a, b]$ into n subintervals each with length h. Denote the endpoints of the subintervals by x_i for $i = 0 \ldots n$, so $x_i = a + i * h$ and $x_0 = a$, $x_n = b$. Now on each subinterval,the trapezoid with vertices $(x_i, 0)$, $(x_i, f(x_i))$, $(x_{i+1}, f(_{i+1}))$ and $(x_{i+1}, 0)$, see the figure, has area that is approximately equal to the area between the curve and the x-axis from x_i to x_{i+1}.

Thus the total area between the curve and the x-axis from a to b is approximately equal to the sum of the area of the trapezoidal areas. That is, the quantity

$$\sum_{i=1}^{n} (f(x_{i-1}) + f(x_i))\frac{h}{2} = \frac{b-a}{2n}\left(f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right)$$

is approximately equal to the area between the curve and the x-axis from a to b.

It is easy to write an algorithm for calculating the trapezoidal approximation to the area that could be programmed on a computer. Here is the statement of the algorithm:

Algorithm(Trapezoid Rule)

- Input: The left endpoint a, the right endpoint b, the number of subintervals n, and the function $y = f(x)$.
- area $= (f(a) + f(b))/2$
- For i=1 to $n - 1$

  area $=$ area $+ f(a + i\frac{b-a}{n})$

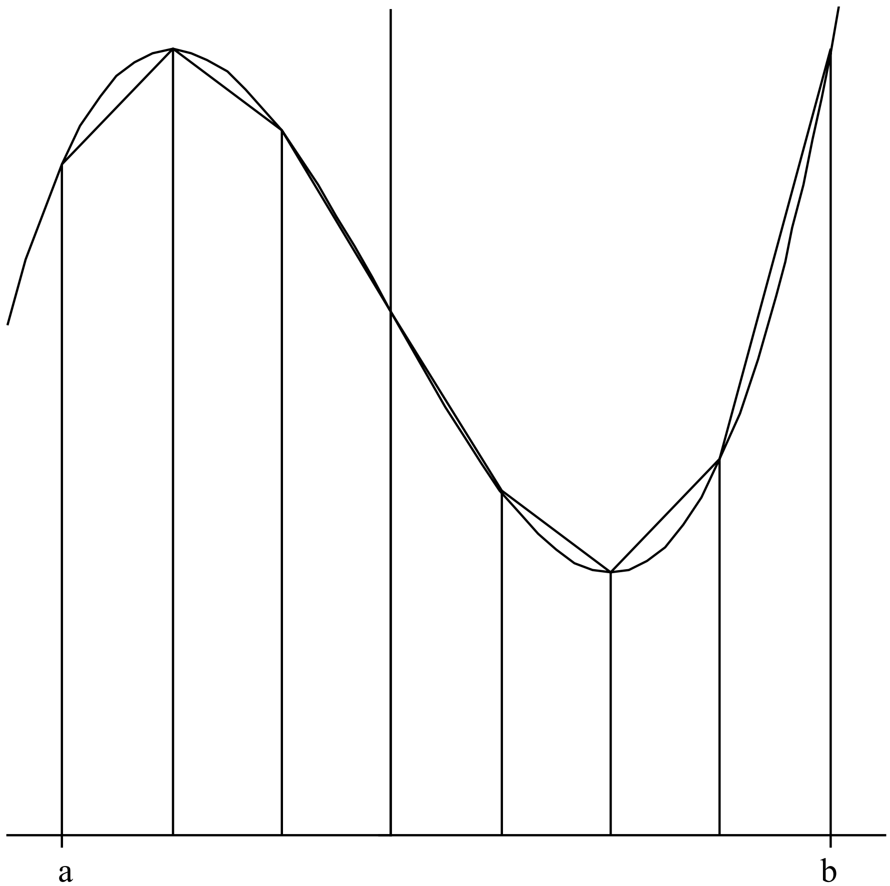


FIGURE 5.3. Trapezoidal Approximation

- area $= \left(\frac{b-a}{n}\right)$ area

Let's write a program for the TI-89 does the calculations in this algorithm. The program can be used for any function which is entered as `y1(x)`. The computed result will be an approximation to the area under the curve and above the interval on x-axis from the left endpoint a to the right endpoint b. When calling the program we will list the left and right endpoints of the interval, and the number of subintervals to be used in the approximation.

```
:  trap(a,b,c)
:  (y1(a)+y1(b))/2.→area
:  (b-a)/n→h
:  For i,1,n-1,1
:  y1(a+i*h)+area→area
:  EndFor
:  area*h→area
```

```
:  Disp area
:  EndPrgm
```

Several things about this code should be mentioned. First, entering "2." rather than simply "2" was intentional. Because if "2" is entered the TI-89 might do all calculations with exact arithmetic. This can result in an area approximation in the form of a fraction with a very large numerator and denominator. The calculations will be lengthy and time consuming, and usually one is more interested in a decimal form for the approximation. Second, to avoid computing $\frac{b-a}{n}$ every time it appears in the algorithm, it is computed once and stored in h.

It's interesting to compare the trapezoidal approximation to the area under the parabola $y = x^2$ with the inner and out rectangle approximations. Suppose we take $n = 20$ in both cases. The Trapezoid Rule program gives the approximate area as .33375. The formulas above give an outer rectangle approximation of .35875 and an inner rectangle approximation of .30875. Thus, at least in this case, the trapezoidal approximation is more accurate. For smooth boundaries, this is generally the case. Notice also that in this case the approximation given by the Trapezoid Rule is exactly the average of the outer and inner rectangle approximations. Will this always be the case?

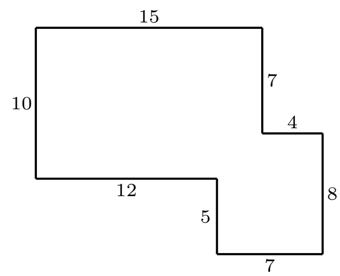
Nonetheless, in one respect, the method of Archimedes for the area under a parabola has an advantage over the trapezoidal approximation. That is, there were two rectangular approximations, one that was larger than the area under the parabola and one that was smaller. One could thus infer the accuracy of the approximation. That nice situation continues to hold whenever the rectangle method is used with an *increasing* function. For the trapezoidal approximation, we see from the figure that on some subintervals the approximations are larger than the actual area and on some they are smaller. Thus, we have no such simple way of assessing the accuracy of the trapezoidal approximation. Indeed, one might wonder if it could happen that when adding all the areas over the subintervals the errors accumulate in a way that gives a very poor approximation or that as the number of subintervals increases the approximations do not improve. Although it is beyond the scope of this workshop, it can be shown that for continuous functions $y = f(x)$ the trapezoidal approximations do approach the true value of the area as the number of subintervals increase.

CHAPTER 6

Problem Solving with Algebra

1. Area Problems

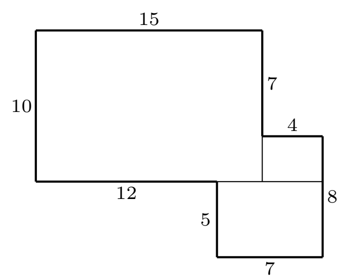
Consider the following geometric figure constructed from two overlapping rectangles. We would like to compute its area and perimeter. Our first task would be to measure its sides. We have written in its measurements for you.



With these measurements, it is easy to compute the perimeter,

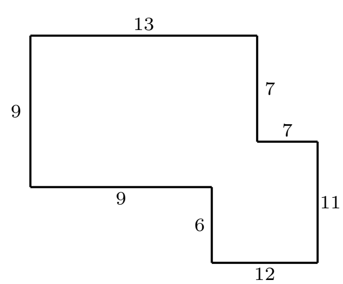
$$15 + 7 + 4 + 8 + 7 + 5 + 12 + 10 = 68;$$

but, computing the are is not so easy. One way to do this is to cut the region up into rectangles that do no overlap. There several ways to do this; we picture one below:

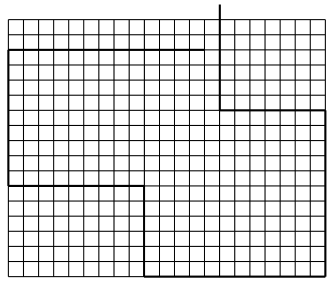


The area then is $A = 10 \times 15 + 5 \times 7 + 3 \times 4 = 197$.

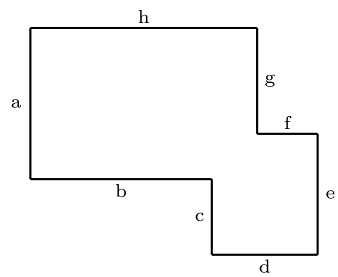
Now consider another overlapping rectangle region with a different set of measurements:



If we try to draw this figure on a grid, we see that it does not close up! What's wrong?



We can use our algebra to better understand this figure. First, we replace all lengths by variables:

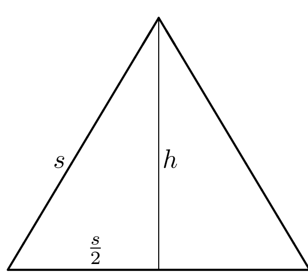


The first question we ask is: How wide is this region from east to west? Measuring along the top or north sides we have $f + h$ for the width ; but measuring along the south sides, we get $b + d$. So one condition that must be met for the boundary to close is $b+d = f+h$. Measuring the region from north to south, we get $e+g$ from the east and $a + c$ from the west giving the second equation $a + c = e + g$.

Hence, if we are given any three of east-west measurements (b, d, f, h), then we can compute the fourth. And, if we are given any three of north-south measurements (a, c, e, g), then we can compute the fourth. If we delete the two 7 unit measurements in the above example, we compute the correct east-west measurement to be $8 = 9+12-13$ and the correct north-south measurement to be $4 = 9+6-11$, Using these measurements, the drawing on the grid closes.

2. More Formulas

Consider an equilateral triangle with side length s:



$$P = 3s$$
$$A = \frac{sh}{2}$$

Here we have four variables: s, side length; h height; P, perimeter and A area. They are not independent. In fact, you can fix any one of them and then the other three are determined. Our first task is to compute h in terms of s. By the

Pythagorean Theorem $s^2 = (\frac{s}{2})^2 + h^2$. Multiplying through by 4, collecting terms, dividing through by 4 and taking the square root of both sides:

$$\begin{aligned} (\tfrac{s}{2})^2 + h^2 &= s^2 \\ s^2 + 4h^2 &= 4s^2 \\ 4h^2 &= 3s^2 \\ h^2 &= \tfrac{3}{4}s^2 \\ h &= \tfrac{\sqrt{3}}{2}s \end{aligned}$$

Using this equation we can solve for the area in terms of the side length:

$$A = \frac{sh}{2} = \frac{1}{2}s\frac{\sqrt{3}}{2}s = \frac{\sqrt{3}}{4}s^2.$$

Since $s = \frac{P}{3}$, we then have:

$$A = \frac{\sqrt{3}}{36}P^2.$$

Exercise 6.1. Find the side length of the equilateral triangle that has the number of units in its perimeter equal to the number of square units in its area.

[[Work out formulas for isosceles triangle with base b, height h, side length s, area A and perimeter P.]]

[[Work out formulas for symmetric trapezoid with base b, height h, top length t, area A and perimeter P.]]

CHAPTER 7

Covering and Surrounding

This chapter was designed as a stand-alone workshop, and can be used as such or as a chapter of the text.

1. Relationships in the Bumper Car Problem

Investigation 1 in *Covering and Surrounding* deals with designing bumper car layouts. Some of the problems deal with finding the maximum perimeter given a certain number of blocks. The book mentions that the largest perimeter you can get is by putting the blocks in a straight line, but it is not intuitively clear why this solution is the best.

For example, if you want to find a design with maximum perimeter on 4 blocks, lining up the blocks in a straight line gives you a perimeter of 10, as in Figure 7.1:

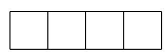
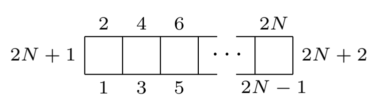

(A)

FIGURE 7.1.

You can get 10 by counting the top and bottom edge of each block and then adding 2 for the edges on the left and right. Thus, if you line up N blocks in a line, the perimeter P can be found by $P = 2N + 2$.



(B)

FIGURE 7.2.

The previous example of perimeter 10 on 4 blocks is not the only possible design. The 2 designs on 4 blocks shown in Figure 7.3 also have 10 as the perimeter:

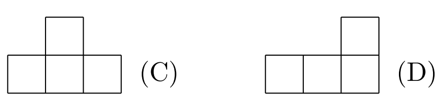


FIGURE 7.3.

Is there a formula to figure out the perimeter based on how the blocks are lined up? Yes! Before we see what the formula is, let's look at one more thing. We'll call a point in our design an *internal point* if 4 blocks meets at that point. For example, in the design in Figure 7.4 there is exactly 1 internal point, while designs (A), (B), (C) and (D) have no interior points.

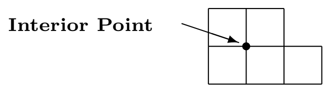


FIGURE 7.4.

A relationship between the number of blocks, N, the perimeter P and the number of internal blocks I is given by $P = 2N + 2 - 2I$. (This formula was derived from a theorem by Euler often used in graph theory.) From this formula, it is evident that P takes on its maximum value when I = 0. Thus, any way you can arrange N blocks without any interior points will give you a maximum perimeter. The simplest way to create such a design is line up all the blocks in a straight line!

2. Constructing Grid Regions

2.1. Basic Principle: All shapes can be constructed from one square by adding squares one at a time and this can be done in three ways, shown in Figure 7.5, Figure 7.6, and Figure 7.7.

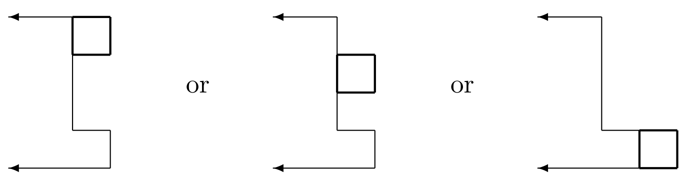


FIGURE 7.5. Add a square touching along one edge: area increases by 1; perimeter increases by 2.

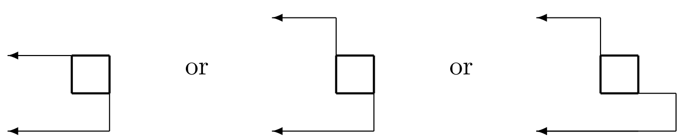


FIGURE 7.6. Add a square in an inside corner: area increases by 1; perimeter stays the same.

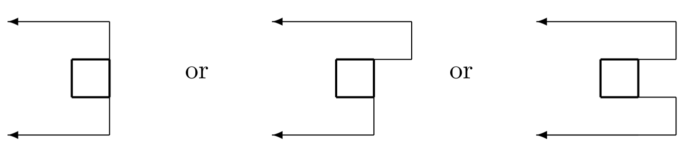


FIGURE 7.7. Add a square in a slot: area increases by 1; perimeter decreases by 2.

2.2. What can we conclude from this?

- Perimeter is always an even number.
- A region with perimeter is as large as possible for a given area if you never fill an inside corner or slot when constructing it. Then $P = 2A + 2$.
- A region with area as large as possible for a given perimeter has no inside corners.
- Area that is as large as possible for a given perimeter is a rectangle.
- Area that is as large as possible for a given perimeter is square or a rectangle with one side just one unit longer than the other. Then $A = \frac{P^2}{16}$ or $A = \left(\frac{P+2}{4}\right)\left(\frac{P-2}{4}\right) = \frac{P^2}{16} - \frac{1}{4}$
- We can always avoid filling a slot in constructing a region.
- $P = 2(A + 1 - I)$, where I is the number of interior points.

3. How many pentominos are there?

Investigation 3 in *Covering and Surrounding* describes a *pentomino*. A *pentomino* is an arrangement of five unit squares that are joined along their edges. Up to isomorphism (turning or flipping), there are only 12 pentominos. Let's investigate why.

First, notice that with one square there is only one design, as shown in Figure 7.8.

(I)

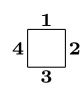


FIGURE 7.8.

Now, adding a square to 1, 2, 3, or 4 would all be equivalent, so there is only one design with 2 squares, shown in Figure 7.9.

(II)

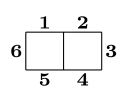


FIGURE 7.9.

Notice that in design (II), adding a block to 3 is the same as adding a block to 6. Also, adding a block to 1 is equivalent to adding a block to 2, 4, or 5. So there are exactly 2 designs with three blocks, shown in Figure 7.10.

(III)

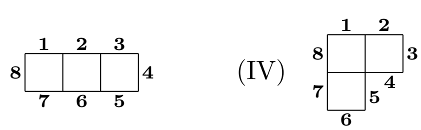


FIGURE 7.10.

To find all designs with 4 blocks (tetrominos), we add a block to each of the edges 1-8 in (III) and (IV) and throw out duplicates. In design (III) adding a block to 1, 3, 5, or 7 all yield the same design. Two other designs can be formed by adding a block to 4 or 8 and 2 or 6, and all three are shown in Figure 7.11.

You can still get other tetrominos by adding blocks to design (IV). However, some of these will be repeat designs. For instance, adding a block to 1 or 8 in design

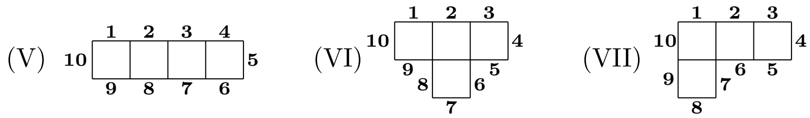


FIGURE 7.11.

(IV) gives us design (VI), while adding a block to 3 or 6 results in design (VII). Two new tetrominos can be obtained to by adding blocks to 2 or 7, and 4-5, shown in Figure 7.12.

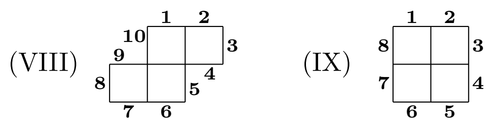


FIGURE 7.12.

Now, all possible tetrominos are depicted in designs (V)-(IX). To find the total number of pentominos, we can add a block to each edge in designs (V)-(IX) and get rid of any duplicates.

The 12 possible pentominos are shown in Figure 7.13.

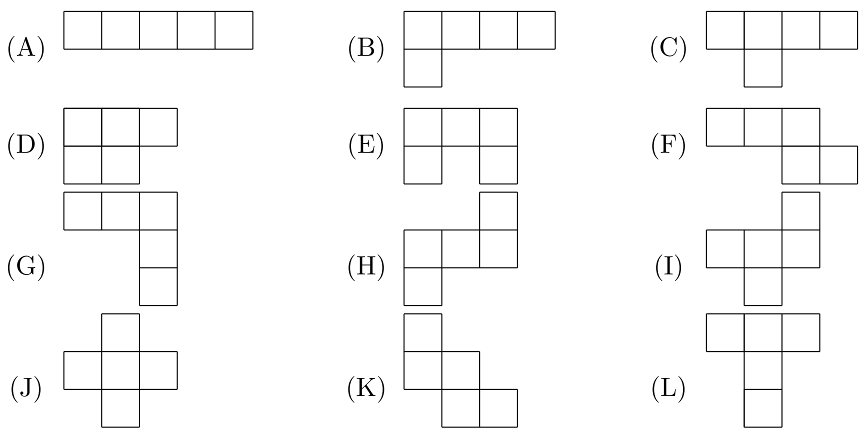


FIGURE 7.13. The 12 Pentominos

Each of these pentominos was created by adding a block to one of the tetrominos:

- Adding a block to edges 5 or 10 of (V) results in pentomino A.
- Adding a block to edges 1, 4, 6, or 9 of (V) or 4 of (VII) results in pentomino B.
- Adding a block to edges 2, 3, 7, or 8 of (V) or 4 or 10 of (VI) or 10 of (VII) results in pentomino C.
- Adding a block to edges 5-6 or 8-9 of (VI) or 6-7 of (VII) or 4-5 or 9-10 of (VIII) or any edge of (IX) results in pentomino D.
- Adding a block to edge 5 of (VII) results in pentomino E.
- Adding a block to edges 9 of (VII) or 3 or 8 of (VIII) results in pentomino F.
- Adding a block to edge 8 of (VII) results in pentomino G.

- Adding a block to edge 3 of (VII) results in pentomino H.
- Adding a block to edges 2 of (VII) or 1 or 3 of (V) or 1 or 6 of (VIII) results in pentomino I.
- Adding a block to edge 2 of (VI) results in pentomino J.
- Adding a block to edges 2 or 7 of (VIII) results in pentomino K.
- Adding a block to edges 7 of (VI) or 1 of (VII) results in pentomino L.

This systematic approach shows that there are exactly 12 different pentominos!

4. Challenging Exercises

EXERCISE 7.1. Consider the two Regions depicted in Figure 7.14.

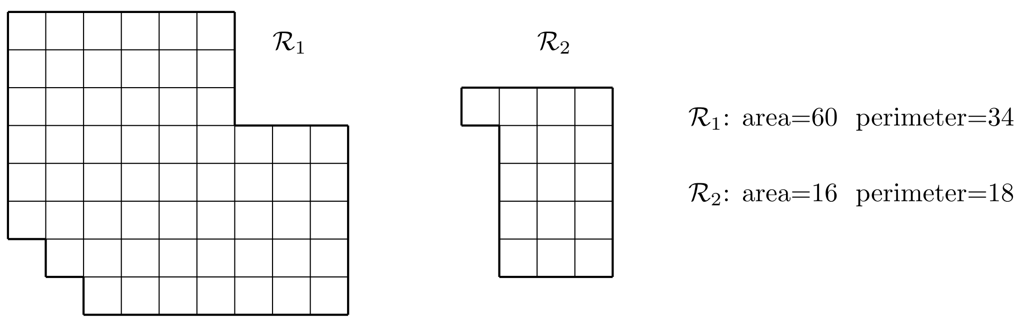


FIGURE 7.14.

- Without recounting, compute the area of the region obtained by abutting these regions as shown in Figure 7.15

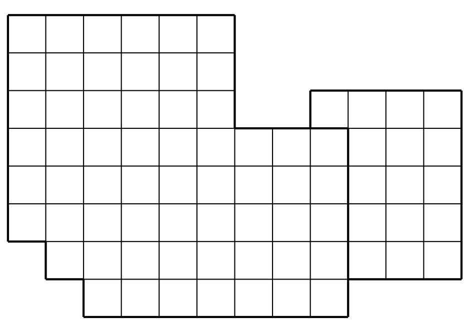


FIGURE 7.15.

- Without recounting, compute the area of the region obtained by overlapping the regions as in Figure 7.16
- Without recounting, compute the area of the region obtained by removing a copy of the smaller region from the larger region, as shown in Figure 7.17.

EXERCISE 7.2. Consider any two regions (not necessarily grid regions) $\mathcal{R}_1$ and $\mathcal{R}_2$ with areas A_1, A_2 and perimeter P_1, P_2.

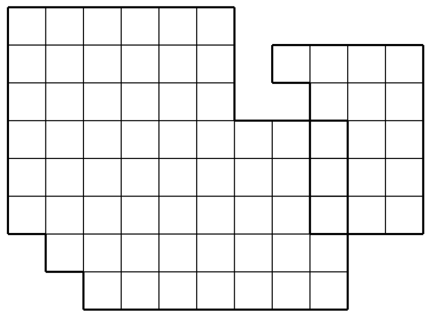


FIGURE 7.16.

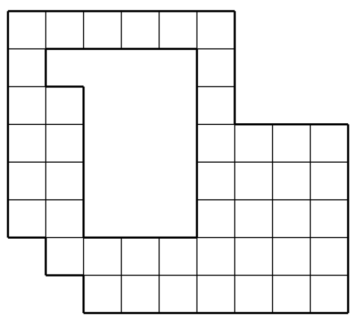


FIGURE 7.17.

- Give formulas for the area and perimeter of the region formed by abutting $\mathcal{R}_1$ and $\mathcal{R}_2$ along a boundary of length N.
- Give formulas for the area and perimeter of the region formed by overlapping $\mathcal{R}_1$ and $\mathcal{R}_2$ ($\mathcal{R}_1 \cup \mathcal{R}_2$), when the region common to both ($\mathcal{R}_1 \cap \mathcal{R}_2$) has area A_0 and perimeter P_0.
- Give formulas for the area and perimeter of the region formed by removing the region $\mathcal{R}_2$ from the interior of $\mathcal{R}_1$.

EXERCISE 7.3. Consider any grid region $\mathcal{R}$. Its boundary has *inside corners* and *outside corners*, as shown in Figure 7.18.

Let IC denote the number of inside corners and OC denote the number of outside corners. Give an equation relating IC and OC Check your equation on several different regions. Try to formulate an explanation for this equation.

EXERCISE 7.4. A grid region $\mathcal{R}$ can have several kinds of symmetries: mirror symmetries, half-turn symmetries, and quarter-turn symmetries.

- List all symmetries for each of the regions in Figure 7.19.
- Construct regions that exhibit each possible kind of symmetry. And then construct regions that exhibit each possible collection of symmetries.

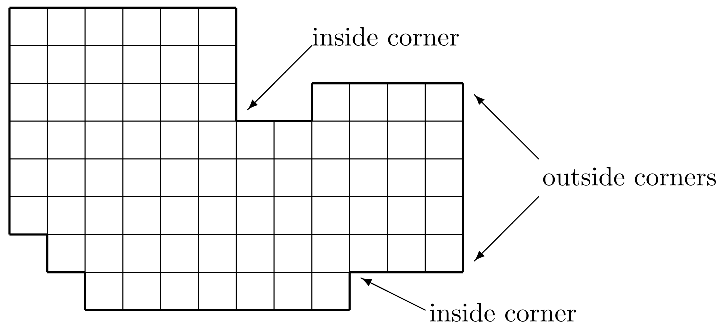


FIGURE 7.18.

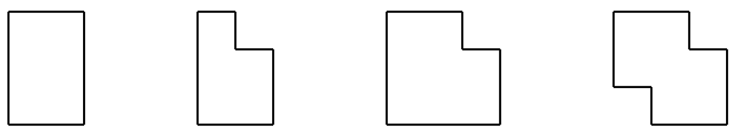


FIGURE 7.19.

5. Can We Estimate Perimeter Using a Grid?

Investigation 2 in *Covering and Surrounding* deals with measuring odd shapes. A teacher's note (p. 28b) describes how using smaller grids can help students better estimate the area. One might conclude that estimating perimeter can be done in the same way. Let's investigate this thought. Consider the right triangle shown in Figure 7.20 that has legs of size 1 unit:

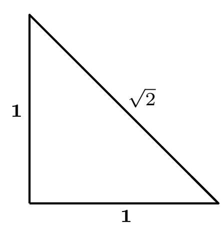


FIGURE 7.20.

The perimeter of this triangle is $2 + \sqrt{2} \approx 3.414$ units. If we estimate the perimeter with blocks of size $\frac{1}{4}$ we have $P \approx 3(\frac{1}{4}) + 3(\frac{1}{4}) + 6(\frac{1}{4}) = 3$. A graphical representation of this is shown below, in Figure 7.21:

Now, if we try to estimate the perimeter of the triangle with blocks of size $\frac{1}{8}$ we have $P \approx 7(\frac{1}{8}) + 7(\frac{1}{8}) + 14(\frac{1}{8}) = \frac{28}{8} = 3.5$, as shown in Figure 7.22.

So when we use blocks of size $\frac{1}{8}$ we get really close to the actual perimeter. But notice that our answer is slightly larger than the perimeter. If we were to use

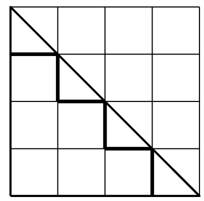


Figure 7.21.

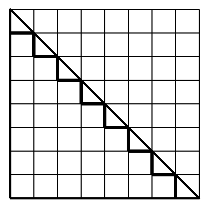


Figure 7.22.

blocks of size $\frac{1}{n}$, our estimated perimeter would be:

$$P \approx \frac{n-1}{n} + \frac{n-1}{n} + 2\left(\frac{n-1}{n}\right) = \frac{4n-4}{n} = 4 - \frac{4}{n}.$$

So as we pick larger and larger n (thus using smaller and smaller blocks) the perimeter we get will be close to 4 which is not the actual perimeter at all! Therefore, estimating perimeter using a grid can lead to false conclusions.

6. The "Toothpick Formula."

One investigation in *Covering and Surrounding* deals with counting the number of edges in a grid region or the number of toothpicks needed to construct the region. Specifically it considered the sequence of grid regions shown in Figure 7.23:

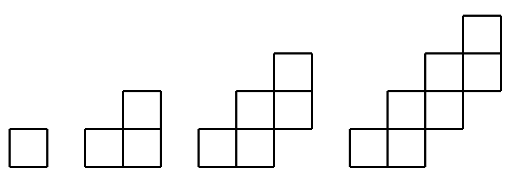


Figure 7.23.

The numbers of toothpicks needed to construct these are 4,10,16 and 22. The student is to observe that the number is increased by 6 at each step.

The general formula for the number of toothpicks is quite simple:

$$T = 2A + \frac{1}{2}P,$$

where T is the number of toothpicks, A, the area and P, the perimeter. Furthermore, it is very easy to explain this formula.

We will derive the formula by counting each toothpick twice (once from each side) and then divide the total by 2. Standing inside a square, we count 4 toothpicks in its boundary. So $4A$ counts each toothpick in the interior twice, once from each side, and each toothpick on the perimeter once, from the inside. So $4A + P$ counts each toothpick exactly twice and dividing by 2 gives the formula.

We can combine this formula with the formula for perimeter to get a second toothpick formula in terms of area and the number of interior points:

$$T = 3A + 1 - I.$$

To combine $T = 2A + \frac{1}{2}P$ and $P = 2A + 2 - 2I$, divide the second equation by 2 to get

$$\frac{1}{2}P = A + 1 - I$$

then replace $\frac{1}{2}P$ by $A + 1 - I$ in the first formula to get

$$T = 2A + A + 1 - I = 3A + 1 - I.$$

Covering and Surrounding was originally written by Jack Graver, Lawrence Lardy, and Christina Niemerg, as a BAMA Workshop in March 2005

CHAPTER 8

Looking Down the Line

1. Basic Examples

Algebraically, a linear function is one which reduces to the form $f(x) = ax + b$ where a and b are real constants. Preferably, we consider the case where $a \neq 0$, as $f(x) = b$ is constant and therefore mostly uninteresting. We work with linear functions quite frequently, though most often we don't stop to consider that they in fact are linear.

Let's consider a simple, common linear function: the total cost of a monthly phone bill. For the sake of simplicity, we'll ignore the taxes and fees applied to the bill, and simply look at the basic costs. The simplest cell phone plans are the prepaid plans; on these plans, you purchase in advance a certain number of minutes of usage. Let's say that a particular plan allows you to purchase minutes at a rate of \$20 per 100 minutes. What is cost of the plan?

Since the minutes must be purchased in allotments of 100, we'll say that h is the number of 100-minute blocks we've purchased, and $C(h)$ is the cost. Then we have $C(h) = 20h$. Simple enough. Even easier, though, if the company simply billed us for each actual minute of use at the same rate. So let's say we pay $20/100 = \$0.20$ per minute of use, m. What is our cost? Well, again, we have $C(m) = 0.20m$. These are very trivial examples.

In the case of a wired phone line, there is often a minimum access fee and then an additional per-call fee for local calls; let's ignore long-distance calling. Say that the line fee is \$30 and the per-call cost is \$0.10. Then if x is the number of calls made in a month, the monthly bill should be $C(x) = 30 + 0.1x$. All of these are clearly examples of linear functions.

We can use linear functions for other purposes than calculating the pre-tax cost of our phone bills. In fact, any application where we know that a quantity is changing from some initial amount by a fixed rate, we have a linear function.

Example 1. We have a 7000 gallon swimming pool in the back yard, and as it is summer, are filling it with our garden hose. If the garden hose can flow at a rate of 8 gallons per minute, how long will it take to fill the pool?

Well, if $g(t)$ is the number of gallons of water in the pool after t minutes, we have $g(t) = 8t$, so we simply solve $7000 = 8t$ for $t = 875$ to see that it will take 875 minutes to fill the pool.

If we want to convert that to hours, we have to use another linear function! Given m minutes, the number of hours they equal is $h = m/60$, or to match the earlier format, $h = \frac{1}{60}m$. So we have that it will take $14\frac{7}{12}$ hours to fill the pool.

Exercise 8.1. Let's say we're filling the same pool, but we don't know that the pool is leaking at a rate of 10 gallons per hour! How long will it take to fill the pool?

1.1. Profit and Loss. A good class of applications of linear functions can be found in business in terms of linear cost and revenue functions. The fixed cost for a business is the cost which they pay no matter what; for instance, the monthly rent on the building. The variable cost is often the per-unit cost of production.

EXAMPLE 2. Lou's Sprocket Company is a small manufacturer of gear assemblies. Their machine shop has a base operating cost of \$20000 per month, and the production cost per assembly is \$15. So the cost function then is $C(q) = 20000 + 15q$, where q is the quantity of assemblies produced.

Revenue functions work similarly; there is most often some fixed price at which a company sells their product. Profit, then is the total revenue less the total cost.

EXAMPLE 3. Lou's Sprocket Company sells their gear assemblies for \$40 each. So their profit function is $P(q) = 40q - (20000 + 15q) = 25q - 20000$. Thus the company has to sell at least 800 units to have non-negative profit.

1.2. Intersections. Any two lines which are not parallel must have a unique point of intersection; this is the Parallel Postulate, a fundamental axiom of Euclidean geometry. So, given two linear functions $f(x) = ax + b$ and $g(x) = cx + d$ with $a \neq c$, we can find the unique point of intersection by finding the x such that $f(x) = g(x)$. Assuming then that $ax + b = cx + d$, we immediately can determine that $x = (d - b)/(a - c)$ when $a \neq c$.

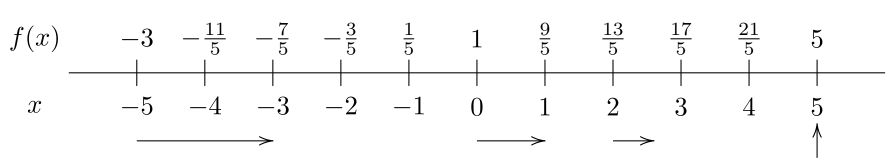
Often this is a good way to determine the more cost-effective of alternatives.

EXAMPLE 4. We are considering buying a house versus renting an apartment, and want to decide how long we must own the home for it to be more cost-effective than continuing to rent. We assume that the house costs \$50000 to purchase and maintenance will amount to \$50 per month on average. On the other hand, to rent an apartment will cost \$750 per month. How long must we live in the house for it to become more cost effective to own the home?

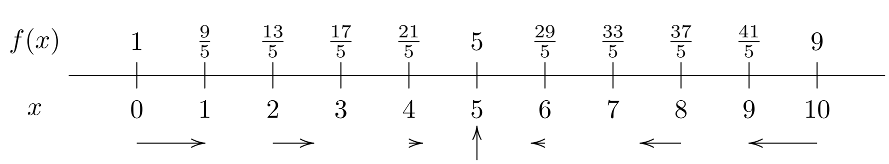
The cost of owning a home for t months can be written $H(t) = 50000 + 50t$; on the other hand, renting for t months costs $R(t) = 850t$. So we set $H(t) = R(t)$ and solve for t: $50000 + 50t = 850t$ gives us $800t = 50000$, or $t = 62.5$. So we must live in the house for just over 5 years for it to be more cost-effective.

2. Geometry of the Linear Function

Consider the function $f(x) = \frac{4}{5}x + 1$. Thinking of it as a transformation of the real line, we may describe it with the following 1-dimensional picture:



Shifting to the right along the number line, we have:



The second viewpoint gives a much more intuitive sense of what is happening geometrically: f is a contraction by a factor of $\frac{4}{5}$ about the number 5 (the center of the contraction). This leads us to the following two questions:

- Does every linear function $f(x) = ax + b$ have a center or fixed point?
- Does every linear function have such a nice geometric description?

Addressing the first question, we must ask what it means for f to have a fixed point. A *fixed point* of a function f is a value x_0 where $f(x_0) = x_0$. So, if $f(x) = ax + b$, we have

$$\begin{aligned} x_0 &= f(x_0) \\ \iff x_0 &= ax_0 + b \\ \iff (1-a)x_0 &= b. \end{aligned}$$

From these we conclude:

LEMMA 1. *Let $f(x) = ax + b$.*

(i) If $a = 1$ and $b = 0$, then f is the identity function, and every point is fixed!
(ii) If $a = 1$ and $b \neq 0$, then f has no fixed points.
(iii) If $a \neq 1$, then $\frac{b}{1-a}$ is the unique fixed point of f.

It is natural then to call $f(x) = x + b$ a *translation*; we write $t_{[b]} = x + b$. If $f(x) = ax+b$ is not a translation, then it has a center $c = \frac{b}{1-a}$, and may be written in *slope-center form*:

$$f(x) = ax + (1-a)c.$$

We write $s_{[a,c]} = ax + (1-a)c$. If $a > 0$ and $a \neq 1$, we call $s_{[a,c]}$ the *dilation* with magnification a and center c. It is natural to call $s_{[-1,c]} = -x + 2c$ a *reflection*.

Now, what if $a < 0$ and $a \neq -1$? Then we have

$$s_{[a,c]} = s_{[-1,c]} \circ s_{[|a|,c]} = s_{[|a|,c]} \circ s_{[-1,c]}.$$

3. Iterating Linear Functions

3.1. Savings and Loans. A particular application of the linear function is that of determining the balance of an interest-bearing account. We will look at a very simple example.

EXAMPLE 5. Suppose we open a savings account at 4.5% interest with an initial deposit of \$1000, and thereafter deposit \$250 per month directly out of our paycheck. Since banks compound interest on a monthly basis, we find that our monthly interest rate will be $\frac{.045}{12} = 0.00375$, or 0.0375%.

Then we can determine the amount in our account after we've made our first deposit of \$250:

$$(1 + 0.00375)1000 + 250 = \$1253.75.$$

Careful observation of the above equation gives us the formula

$$b(x) = (1+i)x + d,$$

where b is the balance of the account at the end of the next month if interest is i, the start-of-month balance is x, and the monthly deposit is d.

Unfortunately, this example only allows us to look one month ahead. What if we know that our interest rate is fixed, and we've set up a direct deposit from our paycheck in a fixed amount every month – can we derive a formula for the amount in account in the n^{th} month? Our formula $b(x) = (1+i)x + d$ is a linear function, so we are essentially asking the question, "What happens when we iterate a linear function?"

This problem is of sufficient general interest to have generated its own terminology. An iterated linear equation is of the form $x_n = ax_{n-1} + b$, called a *linear difference equation*, and the resulting sequence of numbers $x_0, x_1, \ldots, x_n, \ldots$ is called the *solution* to that linear difference equation *with initial value* x_0. What we would really like, though, is a closed-form solution for x_n.

First, though, we look back at the geometric idea of the linear function, and we'll assume that $a \notin \{-1, 0, 1\}$; this allows us to consider the center of the linear difference equation, $c = \frac{b}{1-a}$, and write the following:

$$\begin{aligned} x_n &= ax_{n-1} + b \\ &= ax_{n-1} + (1-a)\frac{b}{1-a} \\ &= ax_{n-1} + (1-a)c \\ &= a(x_{n-1} - c) + c \end{aligned}$$

So then we can actually write $x_n - c = a(x_{n-1} - c)$. But then the right-hand side of the equation is remarkably similar to the left-hand side, and we can write instead $x_n - c = a^n(x_0 - c)$, or equivalently $x_n = a^n(x_0 - c) + c$. So since we started with a and b in our equation instead of c, let's write our equation in terms of those parameters:

$$\begin{aligned} x_n &= a^n(x_0 - c) + c \\ &= a^n\left(x_0 - \frac{b}{1-a}\right) + \frac{b}{1-a} \\ &= a^n x_0 - \frac{a^n b}{1-a} + \frac{b}{1-a} \\ &= a^n x_0 + \frac{(1-a^n)b}{1-a}. \end{aligned}$$

Let's formalize this result.

THEOREM 2. *Let a, b be real numbers with $a \notin \{-1, 0, 1\}$, and let $x_n = ax_{n-1} + b$ be a linear difference equation. Then*

$$x_n = a^n x_0 + \frac{(1-a^n)b}{1-a}.$$

So now we can go back to our example and go a little further.

EXAMPLE 6. (Continued) Remember, we have opened a savings account with an annual interest rate of 4.5% and an initial deposit of \$1000, and have set up our direct deposit to automatically put \$250 into the account every month. So we have

$$x_n = \left(1 + \frac{0.045}{12}\right)x_{n-1} + 250 = 1.00375x_{n-1} + 250,$$

for every integer $n \geq 1$, with $x_0 = 1000$. Using our theorem, we can write this as

$$x_n = 1000(1.00375)^n + \frac{250(1 - 1.00375^n)}{-0.00375}$$

So if we know that our account will be set up in the same way for the next three years, we can determine the balance of the account at that time: three years is 36 months, so we have

$$x_{36} = 1000(1.00375)^{36} + 250\left(\frac{1 - 1.00375^{36}}{-.00375}\right) \approx 10760.77,$$

So after three years, we will have almost $10,800!

This application is relevant to almost everyone; what if, though, we want to specify an amount to have saved in a certain amount of time?

Example 7. Let's say that five years in the future, we want to have $40,000, and our account again pays 4.5%. Again, we open the account with a deposit of $1000. How much do we need to deposit per month in order to reach our goal?

Five years is 60 months; so we know that we want $x_60 \geq 40000$. We also know that $x_0 = 1000$, and $a = 1.00375$. So all that we need to know is b. Solving for b isn't difficult: we know $x_n = a^n x_0 + b(1 - a^n)/(1 - a)$, so since $a \notin \{-1, 0, 1\}$, we have

$$b = (x_n - a^n x_0)\left(\frac{1 - a}{1 - a^n}\right).$$

Substituting the numbers from our example, we get

$$b = (40000 - 1000(1.00375)^{60})\left(\frac{-0.00375}{1 - 1.00375^{60}}\right) \approx 577.08.$$

So if we deposit $577.08 every month for five years, we should have our goal. Let's check:

$$x_n = 1000(1.00375)^{60} + (577.08)\left(\frac{1 - 1.00375^{60}}{-0.00375}\right) \approx 40000.15.$$

Perfect!

This is also the way that loan payments are determined.

Exercise 8.2. We've saved $40,000, and now are using it to pay the down payment on a $200,000 home.

(i) Write a linear difference equation for x_n, the amount of outstanding debt on the loan after n months, using the interest rate i and the monthly loan payment p.
(ii) The bank is offering a 30-year mortgage at 6% interest. How much will the monthly payment be? (Round up to the nearest cent)
(iii) How much will we have actually paid?

Exercise 8.3. Let's look at a different situation: suppose we enter the workforce at the age of 20 and begin saving immediately for retirement at age 65. We open an account at an annual rate of 5.4% compounded monthly, into which we will make regular monthly payments.

(i) If we deposit $100 per month, how much will we have saved for our retirement, to the nearest cent?

(ii) Let's say instead that we deposit \$100 per month for 10 years, after which we realize that our pay has increased so that we can put in \$500 per month. How much will we have upon retirement?

(iii) Now, suppose at at 20 we instead decide that we would like to have save \$1,000,000 by the time we retire. How much should we deposit per month?

3.2. Population Models. Another common application of iterated linear functions is in the study of population models. For example, a certain pond in which trout are not able to reproduce is annually stocked with 2000 trout. If only 2/3 of the trout survive from year to year, at what population will the pond stabilize?

This is clearly a linear model: if x_n is the number of trout in the pond in year n, then we have $x_n = 2000 + 2x_{n-1}/3$. When we use our theorem to determine a closed form for x_n, we get

$$x_n = 2000\left(\frac{1-(2/3)^n}{1/3}\right) = 6000\left(1-\left(\frac{2}{3}\right)^n\right).$$

Now, as n grows to infinity, $\frac{2}{3}^n$ tends toward 0. So the population of the pond will stabilize at 6000 on a long enough timeline.

Exercise 8.4. Given a general exponential function of the form $f(n) = ck^n$, is there some linear difference equation $x_n = ax_{n-1} + b$ such that $f(n) = x_n$?

Exercise 8.5. What are some other common applied models that take this form? What are the underlying linear relationships?

CHAPTER 9

Approximation of Fixed Points

The workshop on linear function presented the *one-line (two-scale) graph* of a linear functions. There we observed that the function $g(x) = \frac{4}{5}x + 1$ maps any point other than 5 to a point closer to 5 and the point 5 is mapped to itself. Thus, the point 5 is called a fixed point of the function $g(x)$. One can take advantage of this observation and repeatedly apply the function to get successive approximations that get closer and closer to the fixed point. Suppose we start with any number other than 5, say $x = 1.5$ to be explicit. The function $g(x)$ maps 1.5 to 2.2. It maps 2.2 to 2.76, and 2.76 to 3.208. If we continue in this way, we obtain a list (or sequence) of numbers each of which is closer to 5 than the previous number. Indeed, we have $|f(1.5) - 5| = |f(1.5) - f(5)| = |\frac{4}{5}(1.5 - 5)| = \frac{4}{5}(3.5)$. In general, the distance from one approximation to 5 is $\frac{4}{5}$ times the previous distance. In other words, the error in the approximation is reduced by a factor of $\frac{4}{5}$ with each application of $g(x)$.

This idea in the method described above for the specific linear function $g(x) = \frac{4}{5}x + 1$ has been used since antiquity, even in ancient Babylon, to find approximate solutions of equations. In fact Heron of Alexandria (who lived around the middle of the first century) has been credited with using it to find $\sqrt{3}$. His reasoning goes something like this: Guess a value for $\sqrt{3}$, call it x_0. If x_0 is less than $\sqrt{3}$, then $\frac{3}{x_0}$ will be greater than $\sqrt{3}$. So $\sqrt{3}$ must lie between x_0 and $\frac{3}{x_0}$. Conversely, if x_0 is greater than $\sqrt{3}$, then $\frac{3}{x_0}$ will be less than $\sqrt{3}$. So again, $\sqrt{3}$ must lie between x_0 and $\frac{3}{x_0}$. Thus, in either case, the average of x_0 and $\frac{3}{x_0}$ is likely to be an even better approximation to $\sqrt{3}$. Therefore, take

$$x_1 = \frac{1}{2}\left(x_0 + \frac{3}{x_0}\right)$$

as the next approximation, and repeat the process. It is worthwhile to state this idea more generally, for any positive number a, as an algorithm.

ALGORITM 1. Let a be a positive number and let x_0 be an approximation to the square root of a.
For $n = 1, 2, \ldots$ calculate

$$x_n = \frac{1}{2}(x_{n-1} + \frac{a}{x_{n-1}})$$

It is easy to do the calculations in this algorithm on the TI-89 calculator. In fact there are several ways to do them, the simplest being the method shown in Figure 9.1. Here we first store the value 3 in `a`, then store 1.5 in `x`, then enter `(x+a/x)/2→x` and press `ENTER`. This calculates the first approximation 1.75 and stores it in `x`. Now each time `ENTER` is pressed, the calculator computes another approximation. The next screen shows the second and third approximations.

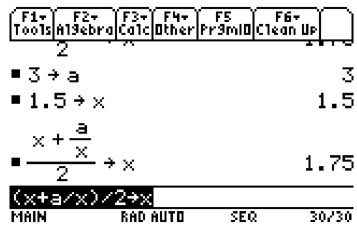


FIGURE 9.1.

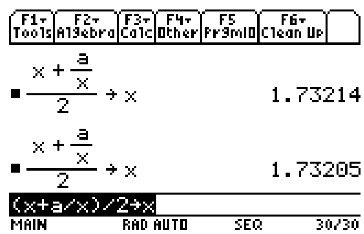


FIGURE 9.2.

This idea of successive approximations can be stated in the following very general form,

ALGORITM 2. Let x_0 be an initial approximate solution to the solution of an equation written in the form $x = g(x)$.
For $n = 1, 2, 3, \ldots$ calculate additional approximations as follows:

$$x_n = g(x_{n-1})$$

Two additional methods for getting the approximations on the TI-89 are now considered. The first method uses the sequence capability of the TI-89 and the second uses a short program. We illustrate the sequence method for the Heron's algorithm and then show how to write a program to implement the general algorithm.

For sequence method, first, store 3 in `a`. Then press `MODE` and select `SEQUENCE` for the `graph` option as shown in Figure 9.3.

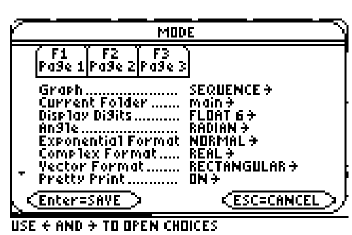


FIGURE 9.3.

Then press ◇ `TblSet` and use the setup values shown in Figure 9.4. Press

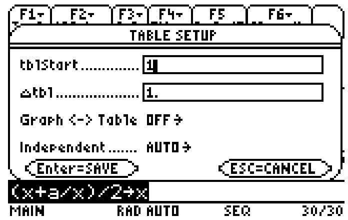


FIGURE 9.4.

♢ `WINDOW` and enter the values shown in Figure 9.5. Press ♢ `Y=` and enter the

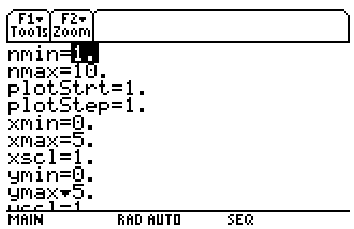


FIGURE 9.5.

expression from the algorithm and the starting approximation shown in Figure 9.6.

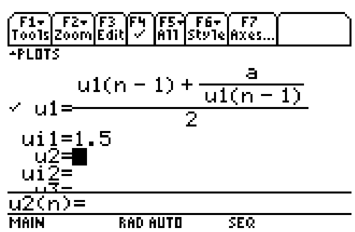


FIGURE 9.6.

Now press ♢ `TABLE` to get the the table of approximations shown in Figure 9.7.

EXERCISE 9.1. Use the method of Heron of Alexandria on a calculator to find accurate approximations to the following square roots.

(i) $\sqrt{16}$, take $x_0 = 2$
(ii) $\sqrt{5}$, take $x_0 = 2$
(iii) $\sqrt{329}$, take $x_0 = 20$

An application of the method of successive approximations was used by some early electronic computers to perform the operation of division approximately while using only the operations of addition, substraction, and multiplication. Since these

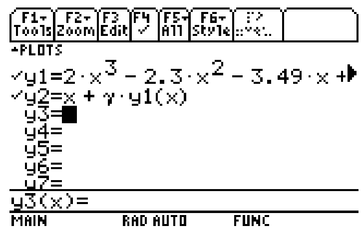


FIGURE 9.7.

computers used base 2 arithmetic, they could of course find the reciprocal of a power of 2 by simply changing the sign of the power. Dividing b by a is equivalent to solving the equation $ax = b$ for x. By multiplying both sides of this equation by a suitable power of 2 we obtain an equivalent equation $a'x = b'$ where $\frac{1}{2} \leq a' < 1$. For example, suppose we want to divide 8 by 2.5. This is equivalent to solving 2.5x=8. If we multiply both sides of this equation by $\frac{1}{2^2} = .25$, we obtain $.625x = 2$. In the general case, we now assume that the equation $ax = b$ has been transformed by multiplying by a power of 2 so that $\frac{1}{2} \leq a \leq 1$. Now rewrite this equation in the form $x = (1 - a)x + b$. Notice that $0 <\leq (1 - a) \leq \frac{1}{2}$. Let x_0 be an initial approximation to $\frac{b}{a}$. For the equation $.625x = 2$ this yields $x = .375x + 2$ When we calculate the successive approximations starting with $x_0 = 2$ we obtain the .275, 3.03125, 3.13672, 3.17627, etc. These calculations are easy to do using the TI-89. First, type `.375*2+2` and press `ENTER`. Next type `*.375+2`, the calculator displays `ans(1)*.375+2`, and press `ENTER`. Now simply press `ENTER` repeatedly to get subsequent approximations.
To see what is going on, consider again the general case in the form $x = (1-a)*x+b$ with starting value x_0. The next approximation is $x_1 = (1 - a)x_0 + b$, and the nth approximation is given by $x_n = (1 - a) * x_{n-1} + b$. Since $x = \frac{b}{a}$ is a solution of $x = (1 - a)x + b$ and since $x_1 = (1 - a)x_0 + b$, when the second of these equations is subtracted from the first we obtain $x - x_1 = (1 - a)(x - x_0)$. Thus, $|x - x_1| = (1 - a)|x - x_0|$, and we see that the new approximation, x_1, has a smaller error by a factor of $(1 - a)$. When we calculate $x_2 = (1 - a)x_1 + b$ similarly find that $|x - x_2| = (1 - a)|x - x_1| = (1 - a)^2|x - x_0|$. That is, the errors in the successive approximations decrease to zero and we can obtain highly accurate approximations to $\frac{b}{a}$ using only the arithmetic operations of addition, substraction, and multiplication.

EXERCISE 9.2. Use your calculator and the above method of successive approximations to solve the following equations without using division.

(i) $.8x = 26$ using 1 as the initial approximation.
(ii) $35x = 261$ using 5 as the initial approximation after transforming the given equation.

Here is a way to view the calculations for solving $ax = b$ with $\frac{1}{2} \leq a < 1$. Since $0 < (1 - a) \leq \frac{1}{2}$, the geometric series $1 + (1 - a) + (1 - a)^2 \ldots$ has the sum $\frac{1}{1-(1-a)} = \frac{1}{a}$. Thus, $\frac{b}{a} = b + b(1 - a) + b(1 - a)^2 + \ldots$. If we take $x_0 = b$ for the method of successive approximations then we $x_1 = (1 - a)b + b$ and $x_2 =$

$(1-a)x_1+b = (1-a)^2b+(1-a)b+b$. Each term of the sequence of approximations is the sum of corresponding number of terms in the geometric series.

EXERCISE 9.3. Let x_0 be an arbitrary initial approximation to the solution of $ax = b$ with $\frac{1}{2} \leq a < 1$. Find algebraic expressions for x_1, x_2, x_3, and the nth approximation x_n. Explain what happens to the term involving x_0 as n gets large.

The ancient method of successive approximations is a powerful idea that is often used even today for finding solutions of equations. However, it is often difficult to write an equation that is of the form $f(x) = 0$ in an equivalent successive approximation form, $x = g(x)$, so that the method gives better and better approximations from step to step. Moreover, the quality of the approximations can change significantly when the form of the iteration function, $g(x)$, in the fixed point equation is changed only slightly. In fact, a poor choice of $g(x)$ can lead to a sequence that moves further away from a solution. The essential requirement for getting better approximations at each step is that the solution be a point of attraction of the fixed point of $g(x)$. That is, points near the fixed point should be mapped by $g(x$ to points closer to the fixed point as in our first example.

Consider the equation

$$2x^3 - 2.3x^2 - 3.49x + 2.07 = 0.$$

This cubic equation has three real roots. One can discover this by plotting a graph or by examining a table of values and noting where the function values change sign. For instance, the function. $f(x) = 2x^3 - 2.3x^2 - 3.49x + 2.07$ has the value -1.72 when $x = 1$ and the value 1.89 when $x = 2$. So we expect to find one or more solutions of the equation between 1 and 2. If we multiply both sides of the equation $f(x) = 0$ by a constant γ and add x to both sides of the result we obtain the equivalent equation

$$x = x + \gamma(2x^3 - 2.3x^2 - 3.49x + 2.07)$$

. This is a form where fixed point iteration can be tried. A good choice for γ is tricky. There are advanced techniques that help guide the choice, but are beyond the scope of this work.

We will now write a general program for successive approximations on the TI-89 and illustrate its use by applying it to obtain a solution to the above equation. The code for the algorithm will input the initial approximation and the number of approximations to be calculated. It is usually not easy to determine in advance how many approximations will be needed to obtain a certain accuracy, so here we'll rely on trial and error. To program the method of successive approximations, begin by pressing `APPS`, in the menu that appears select item `7:Program Editor`, and then choose item `3:New`and press `ENTER`. This gives the screen shown in Figure 9.8.

Scroll down to the `Variable` region and type `sa` followed by `ENTER`. In screen that opens, you will type the program shown below where the `For` statement is obtained by pressing `F2` and choosing item `4:For ...EndFor`.

```
    :sa(x,k)
:Prgm
:For n=1, k,1
:y2(x)→x
:Disp n,x
```

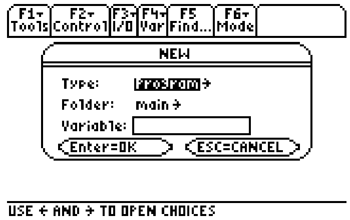


FIGURE 9.8.

```
:EndFor
:EndPrgm
```

Now press `HOME` to return to the home screen. To use this routine, the function $g(x)$ is entered as item `y2` in the `Y=` list of functions. To access this list type ◇ `Y=`. For the specific equation above with $\gamma = -.1$, we first store -.1 in γ by typing `-.1`→`2nd CHAR`, selecting `1:Greek`, then selecting `4:`γ, and finally pressing `ENTER`. Now enter the function $f(x)$ as `y1` and $g(x)$ as `y2` in the `Y=` list as shown in Figure 9.9.

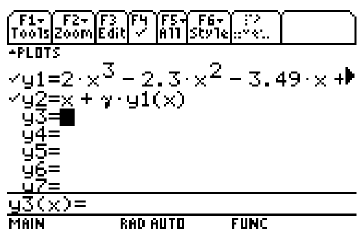


FIGURE 9.9.

Return to the home screen and type `sa(2,6)` (as shown on the first screen below) followed by `ENTER` to get the first six approximations for the initial approximation of 2. The results are shown in Figure 9.10.

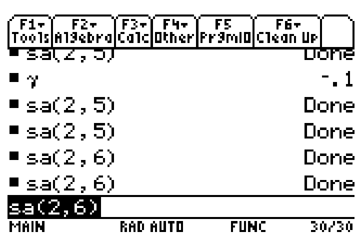


FIGURE 9.10.

The table in Figure 9.11 shows the sequence of successive approximations obtained with $\gamma = -.1$ and $x_0 = 2$.

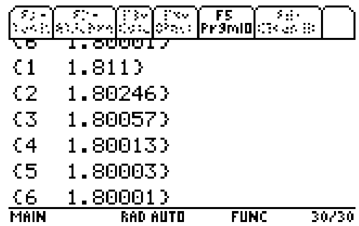


FIGURE 9.11.

EXERCISE 9.4. Find the first 5 successive approximations for the equation

$$x = x + \gamma(2x^3 - 2.3x^2 - 3.49x + 2.07)$$

when:

(i) $\gamma = -.1$ and $x_0 = 1$.
(ii) $\gamma = -.130378$ and $x_0 = 2$.
(iii) $\gamma = .2$ and $x_0 = 1$.
(iv) $\gamma = -.2$ and $x_0 = -1$.
(v) $\gamma = -.2$ and $x_0 = -2$.
(vi) $\gamma = .2$ and $x_0 = -2$.

The trapezoid method for approximating the area of a region is an instance where an approximation is obtained by replacing a function $f(x)$ with a simple linear function over a short interval. That idea can be applied as well to the problem of finding approximations to the solution of an equation $f(x) = 0$. Suppose that two approximations, x_0 and x_1, are known. We can take as a next approximation, x_2 the point where the line joining $(x_0, f(x_0))$ and $(x_1, f(x_1))$ crosses the x-axis. That leads to the following algorithm.

ALGORITM 3 (Secant Method). Let x_0 and x_1 be approximate solutions of the equation $f(x) = 0$.

For $n = 1, 2, 3, \ldots$, calculate

$$x_{n+1} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Here is a TI-89 program for the secant method. It is entered like the code of the successive approximation algorithm. In this case, the first items that are input to the program are the first two approximations to a solution. The third item is the number of additional approximations to be calculated.

```
:secant(a,b,k)
:Prgm
:For n,1,k,1
:(a*y1(b)-b*y1(a))*1./(y1(b)-y1(a))→c
:Disp n,c
:b→a
:c →b
:EndFor
:EndPrgm
```

When the code for the secant method is applied with $f(x) = 2x^3 - 2.3x^2 - 3.49x + 2.07$ using $x_0 = 1.5$ and $x_1 = 2$, the approximations obtained are shown in Figure 9.12.

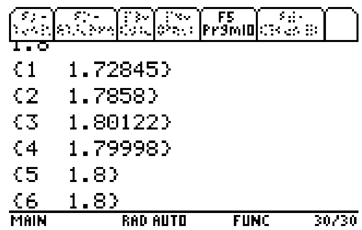


FIGURE 9.12.

EXERCISE 9.5. "Deciding where to put the frets on a guitar depends, in effect, on finding an approximate construction of the twelfth root of 2."(Ian Stewart, *Faggot's fretful fiaso*, in Music and Mathematics: From Pythagoras to Fractals, Edited by John Fauvel, Raymond Flood, and Robin Wilson, Oxford University Press, 2003) Since $1^12 = 1$ and $2^12 = 4096$, the twelfth root of 2 lies between 1 and 2, but will be much closer to 1. Use the `secant` code and your TI-89 to find the approximate value of the twelfth root of 1.

We have seen that the method of successive approximations is a useful tool for finding approximations to solutions of equations with one unknown. Could it be used to approximate the solution of two equations with two unknowns? To be specific, consider the supply and demand equations

$$\begin{aligned} x + 2y &= 900 \\ 2x - 3y &= 50 \end{aligned}$$

The point of stability is a value for x and a value for y that simultaneously satisfies both equations. Of course this solution can be found quite easily, but the question we want answered is: Can these equations be rewritten in an equivalent form so that successive approximations will give accurate approximations to the solution? For the case of a single linear equation in one unknown, $ax = b$, we saw how to transform the equation to an equivalent equation $a'x = b'$ where $\frac{1}{2} \le a' < 1$. Then this equation can be written as $x = (1 - a')x + b'$ and successive approximations led to accurate approximations. Now, we have two equations and two unknowns, so we want to obtain two approximations at each step and we want both of the errors to decrease. Let's suppose that the system

$$\begin{aligned} ax + by &= g \\ cx + dy &= h \end{aligned}$$

can be transformed to the equivalent system

$$\begin{aligned} a'x + b'y &= g' \\ c'x + c'y &= h' \end{aligned}$$

where $\frac{3}{4} \le a' < 1$, $|b'| < \frac{1}{4}$, $|c'| < \frac{1}{4}$, and $\frac{3}{4} \le d' < 1$. Then when successive approximation is applied to

$$\begin{aligned} x &= (1-a')x - b'y + g' \\ y &= -c'x + (1-d')y + h' \end{aligned}$$

the errors satisfy

$$\begin{aligned} x_{n+1} - x &= (1-a')(x_n - x) - b'(y_n - y) \\ y_{n+1} - y &= -c'(x_n - x) + (1-d')(y_n - y) \end{aligned}$$

Since $0 < 1 - a' \le \frac{1}{4}$ and $0 < 1 - d' \le \frac{1}{4}$ we have

$$\begin{aligned} |x_{n+1} - x| &\le \frac{1}{4}|x_n - x| + \frac{1}{4}|y_n - y| \\ |y_{n+1} - y| &\le \frac{1}{4}|x_n - x| + \frac{1}{4}|y_n - y| \end{aligned}$$

It is not obvious how to transform a given system to an equivalent one where the coefficients satisfy the conditions specified above. There are four coefficients to modify and to obtain an equivalent system. We can multiply the first equation by any non-zero constant, m_{11} and add to it a multiple m_{12} of the second equation. Similarly, we multiply the second equation by a non-zero multiply m_{22} and add to it a multiple m_{21} of the first equation. With these four multiples, we can carry out the transformation. This is illustrated in the following example.

Example 8.

Let's return to the system of demand and supply equations considered above. The equivalent system becomes

$$\begin{aligned} (m_{11} + 2m_{12})x + (2m_{11} - 3m_{12})y &= m_{11}g + m_{12}h \\ (m_{21} + 2m_{22})x + (2m_{21} - 3m_{22})y &= m_{21}g + m_{22}h \end{aligned}$$

Thus, we want to choose values of m_{11} and m_{12} such that

$$\frac{3}{4} \le m_{11} + 2m_{12} \le 1 \text{ and } |2m_{11} - 3m_{12}| \le \frac{1}{4}$$

and values of m_{21} and m_{22} such that

$$|m_{21} + 2m_{22}| \le \frac{1}{4} \text{ and } \frac{3}{4} \le 2m_{21} - 3m_{22} \le 1.$$

By considering contour plots of $u + 2v$ and $2u - 3v$ we find that the inequalities are satisfied by taking

$$m_{11} = 0.8,\ m_{12} = .2,\ m_{21} = .1,\ m_{22} = .9.$$

Thus the iteration equations are:

$$\begin{aligned} x_{n+1} &= .2x_n - .2y_n + 370 \\ y_{n+1} &= -.1x_n + .1y_n + 265 \end{aligned}$$

Using the starting values of $x_0 = 300$ and $y_0 = 300$ the first five computed approximations are:

n	x_n	y_n
0	300.	300.
1	370.	265.
2	391.0	254.5
3	397.30	251.35
4	399.190	250.405
5	399.7570	250.1215

Notice that the values of x_n are getting closer to 400, while those of y_n are getting closer 250. The exact solution of the system is $x = 400$ and $y = 250$. Examine the errors in each of the approximations given above.

Exercise 9.6. Using the iteration equations in the example calculate the first five approximations using the indicated starting values.

(i) $x_0 = 0,\ \ y_0 = 0$
(ii) $x_0 = -100,\ \ y_0 = 800$

Can one obtain accurate approximations using any starting values? Why?

Exercise 9.7. Another choice for the multipliers m_{ij} is:

$$m_{11} = .429,\ m_{12} = .286,\ m_{21} = .286,\ m_{22} = -.143$$

(i) Find the iteration equations for this choice of multipliers.
(ii) Calculate the first five approximations using the starting values in the previous exercise.

We have seen that there are many choices for the multipliers that will yield to accurate approximations for the solution of this linear system. Is one choice better than another? If one choice gives accurate approximations in fewer steps than another, it presumably would be preferred. In this sense, the choice—

$$m_{11} = \frac{4}{7},\ m_{12} = \frac{2}{7},\ m_{21} = \frac{2}{7},\ m_{22} = -\frac{1}{7}$$

is as good as it gets, since it yields the iteration equations

$$\begin{aligned} x_n &= 400 \\ y_n &= 265 \end{aligned}$$

These equations give the exact solution at each iteration with out regard to the initial approximation! Notice that these multiples arise from the standard method of elimination for solving the system of equations.

We explored the possibility of using the method of successive approximations to get approximate solutions to a system of two linear equations simply because the question arose naturally in the context of our study of successive approximations. When questions occur naturally to you, explore them! It's fun to do so, and very satisfying to get an answer to a question. Mathematicians do this frequently when exploring the frontiers of a subject. When found that the method of successive approximations could be used to solve the linear system, did we discover a better method? Well, the last observation for a set of multipliers might suggest that the elimination method gives the best multipliers so why not just use them and toss our findings in the waste basket. But we can't foresee all possible applications. For instance, maybe our approach could lead to techniques for solving larger systems approximately, say 100 or more equations. Or there might be an application where systems with nearly the same equations must be solved many times. Since the equations are nearly the same, a single set of multipliers could determined and used each time a system needs to be solved. Moreover, although the simple linear system we examined can be easily solved, there are non-linear systems that can be difficult to solve. The method of successive approximations can be used for these systems as well, and what we learned from studying this simple case might give us insight for this more difficult problem.

CHAPTER 10

Algebra Counts!

1. Introduction to Counting

Counting problems have many applications in mathematics and in many applications of mathematics. For example, within mathematics the ability to count complicated sets of objects is essential to finite probability (e.g how many of all possible 5-card hands are full houses?). For an example outside of mathematics, a typical counting problem from computer science might go like this; how many strings of n 0's and 1's have no repeated 0s? One of the most powerful counting techniques involves the algebra of polynomials and it extension to infinite polynomials or formal power series. But before we get to the algebra, we review the basic counting formulas.

The simplest counting problems are those involving ordering or selecting elements from a set. First off consider n people; how many different ways can they line up? There are n choices for the person at the head of the line then $(n-1)$ choices for the second in line. So we can fill the first two positions in the line in $n \times (n-1)$ different ways. The first three positions can then be filled in $n \times (n-1)$ times $(n-2)$ different ways. And so on. So the entire group can be lined up in exactly $n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$ different ways. We have a short hand notation for this number, the product of all natural numbers from n down to 1: $n!$, n *factorial* in words. The convention is to interpret 0! as 1 not 0.

Now suppose there are n people in candidacy for r distinct positions where $r \leq n$. An assignment of r of the candidates to the r jobs is called an $r-$ *permutation*. How many different ways can the positions be filled, that is how many r-permutations from an n-set are there? There are n choices of candidates to fill the first position, $n-1$ for the second, and so on, down to $n-(r-1)$ choices of the last position. Thus the number of r-permutations from an n set, $P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$. We can write this in simpler form using *factorial* notation. We have $n! = P(n,r) \times (n-r) \times (-r-1)\ldots 1$ and dividing both sides by $(n-r)!$ gives the formula $P(n,r) = \frac{n!}{(n-r)!}$. Recalling the convention 0!=1, we have $P(n,0) = \frac{n!}{n!} = 1$.

Another simple counting formula is that for the $r-$ *combinations* from an n-set: $C(n,r) = \frac{n!}{r!(n-r)!}$. This is the formula we would use if all of the jobs in the previous example were identical. In this case we are simply choosing a subset of r candidates from the set of n candidates and hiring all of them. It is not too hard to justify this formula using the permutation formula. Let N denote the n job applicants and let S be one possible set of r applicants to hire. Even though the r jobs are identical, designate them job 1, job 2, ..., job r. Now there are $r!$ ways we can assign the candidates in S to these labeled jobs and each such assignment is an r-permutation from N. And of course every r-permutation comes from some

$r-combination$. Looking the other way around, every $r-combination$ comes from exactly $r!$, r-permutations and $C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$. Since there is exactly one subset with zero elements, namely the *empty set* and we see that the convention 0!=1 gives the correct answer: $C(n,0) = \frac{n!}{1\times n!} = 1$.

Thinking of these two formula abstractly let N be any set of n elements. $C(n,r)$ counts the number of subsets of N that contain exactly r elements while $P(n,r)$ counts the number of ways that you can select r elements one at a time in order. The two features that are relevant are: there are no repeats allowed in the selection (both cases) and whether the selection is ordered (permutations) or not (combinations).

Exercise 10.1. Let the set N consist of the first 5 letters of the alphabet: A, B, C, D, E. Make a list of all 10 subsets of 3 letters. The problem with listing subsets is that, while they are unordered, we must write down the elements in some order. We will avoid writing down the same set twice in different order by listing the elements of each sub set in alphabetic order. Then besides each set of 3 letters, list the 5 additional ways of ordering the letters in that set, there by expanding the list of $C(5,3) = 10$ combinations to a list of $P(5,3) = 60$ permutations.

Now think about using the letters to make words or license plates. Like with permutations, the order makes a difference; but in this case repeats are allowed. These are often called *r-selections* but there is no special notation for them. In terms of set theory: you have a set N of n elements and you select r elements, one at a time; however, when you select an element, you write down the result of your selestion and then return the element to N. So each time you are selecting from the full set and the result is the ordered list of the items you selected. Hence it is easy to count the r-selections: there are n possibilities for the first element in your list, n possibilities for the second element in your list,$\ldots$,n possibilities for the last element in your list. Hence the number of r-selections from an n set is $n \times n \times \ldots n = n^r$.

Exercise 10.2. In the last exercise, you listed all 60 of the length 3-permutations from the set$\{A, B, C, D, E\}$. These are the 3-selections with no repeats. To complete the list of all of the $5^3 = 125$ 3-selections, you must list all of the 3-selections with repeated elements.

We summarize these three formulas in the following chart:

	repeats permitted	no repeats
ordered	n^r	$P(n,r) = \frac{n!}{(n-r)!}$
unordered	**?**	$C(n,r) = \frac{n!}{r!(n-r)!}$

When we summarize in this way it is clear that one formula is missing: the *unordered selections* (with repeats permitted). For an example where this formula would be of use, consider packages of 20 candies in 5 different flavors. Here $n = 5$ is the set of flavors and $r = 20$ is the number of candies chosen. Of course we must permit flavors to be repeated and, since they are all put in a bag, there is no order to this selection. We will derive this missing formula as we develope our algebraic approach to counting.

EXERCISE 10.3. List the unordered 3-selections from the set$\{A, B, C, D, E\}$.

2. The Binomial Theorem

The Binomial Theorem. $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$, where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$.

It follows that the binomial coefficient $\binom{n}{i}$ equals $C(n, i)$ and may be interpreted as the number of i-subsets of an n-set. Before we give a formal proof of the binomial theorem, let's demonstrate this identification by way of an example. Suppose that we had a basket of fruit containing one apple, one banana, one cantaloupe, and one date. Represent each fruit by its first letter and expand the following polynomial:

$$\begin{aligned}(1+a)(1+b)(1+c)(1+d) &= 1 + a + b + c + d \\ &\quad + ab + ac + ad + bc + bd + cd \\ &\quad + abc + abd + acd + bcd + abcd.\end{aligned}$$

Each term of the expansion represents a specific selection of fruit: b represents selecting only the banana, ad represents the selection of the apple and the date, 1 represents selecting no fruit, etc. If all we wish to know is the answer to question "How many ways can we select two pieces of fruit from the basket?," we may replace each individual factor by $(1 + x)$ to get $(1 - x)^4$. Then the coefficient of x^2 in the expansion is the number two letter terms in the expansion above, that is the number of 2-element subsets from our basket. Similarly, the coefficient of x is the number of 1-element subsets, the coefficient of x^3 is the number of 3-element subsets and the coefficient of x^4 is the number of 4-element subsets.

PROOF. We start the proof of the binomial theorem by defining $\binom{n}{i}$ to be $\frac{n!}{i!(n-i)!}$ and then, by direct computation, proving the identity $\binom{n-1}{i-1} + \binom{n-1}{i} = \binom{n}{i}$, for all $0 < i \leq n$:

$$\begin{aligned}\binom{n-1}{i-1} + \binom{n-1}{i} &= \frac{(n-1)!}{(i-1)!(n-i)!} + \frac{(n-i!}{i!(n-1-i)!} \\ &= \frac{i}{i}\frac{(n-1)!}{(i-1)!(n-i)!} + \frac{(n-i)}{(n-i)}\frac{(n-i!}{i!(n-i-1)!} \\ &= \frac{i(n-1)!}{i!(n-i)!} + \frac{(n-i)(n-i!}{i!(n-i)!} \\ &= \frac{i(n-1)! + (n-i)(n-i!}{i!(n-i)!} = \frac{n!}{i!(n-i)!}\end{aligned}$$

Now we are ready to prove the binomial theorem by induction on n. Clearly, it holds for $n = 1$:

$$(1+x)^1 = 1 + x = \binom{1}{0} + \binom{1}{1}x.$$

Now, for $n > 1$, we assume that the binomial theorem has been shown to hold for $n-1$:

$$(1+x)^{n-1} = \binom{n-1}{0} + \binom{n-1}{1}x + \binom{n-1}{2}x^2 + \cdots + \binom{n-1}{n-1}x^{n-1}$$

Then:

$$\begin{aligned}
(1+x)^n &= ((1+x)(1+x)^{n-1} \\
&= (1+x)\left(\binom{n-1}{0} + \binom{n-1}{1}x + \cdots + \binom{n-1}{n-1}x^{n-1}\right) \\
&= \binom{n-1}{0} + \binom{n-1}{1}x + \cdots + \binom{n-1}{n-1}x^{n-1} + \\
&\qquad + \binom{n-1}{0}x + \cdots + \binom{n-1}{n-2}x^{n-1} + \binom{n-1}{n-1}x^n \\
&= \binom{n}{0} + \left(\binom{n-1}{1} + \binom{n-1}{0}\right)x + \cdots + \left(\binom{n-1}{n-1} + \binom{n-1}{n-2}\right)x^{n-1} + \binom{n}{n}x^n \\
&= \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n
\end{aligned}$$

So the result holds for $n = 1$ and, if the result holds for $n-1$, it holds for n. Hence, by induction the result holds for all n. □

We call the polynomial $(1-x)^n$ the *generating function* for the numbers of subsets of an n-set. In general, if we have a sequence of numbers, $C(n,0)$, $C(n,1)$,. . . , $C(n,n)$ in this case, the polynomial with these numbers as coefficients of the powers of x, $C(n,0) + C(n,1)x + \cdots + C(n,n)x^n$ in this case, is called the *generating function* for that sequence of numbers. The binomial theorem states that the generating function for the combinations of n things can be written succinctly at $(1-x)^n$. The remainder of this chapter is devoted to developing algebraic techniques for finding a succinct form for the generating function of a given sequence of numbers.

3. Enumerators and Generating Functions

Let's return to our fruit basket example from the last section:

$$\begin{aligned}
(a^0 + a^1)(b^0 + b^1)(c^0 + c^1)(d^0 + d^1) &= a^0b^0c^0d^0 + a^1b^0c^0d^0 + a^0b^1c^0d^0 + a^0b^0c^1d^0 \\
&+ a^0b^0c^0d^1 + a^1b^1c^0d^0 + a^1b^0c^1d^0 + a^1b^0c^0d^1 \\
&+ a^0b^1c^1d^0 + a^0b^1c^0d^1 + a^0b^0c^1d^1 + a^1b^1c^1d^0 \\
&+ a^1b^1c^0d^1 + a^1b^0c^1d^1 + a^0b^1c^1d^1 + a^1b^1c^1d^1.
\end{aligned}$$

The factor $(a^0 + a^1)$ is called the *apple enumerator*; $(b^0 + b^1)$ the *banana enumerator*; and so on. The expansion is called the *generating function* of this selection problem. To answer the question "How many ways can we select two pieces of fruit from the basket?," we replaced each individual enumerator by $(x^0 + x^1)$ or simply $(1 + x)$ to get the subset generating function $(1-x)^n$.

With this problem to guide us, let's consider a more complicated counting problem. Suppose our basket contains two apples, along with the banana, cantaloupe, and date. If the apples were distinct, for instance one red and one green, we would simply replace the apple enumerator (a^0+a^1) with the red and green enumerators, (r^0+r^1) and (g^0+g^1), and then proceed as above. Thus for the answers to our counting problems, we would look to the expansion of $(x^0+x^1)^5 = (1+x)^5$. However, the situation changes when the apples are identical – for instance, both apples are red. If we were to replace (a^0+a^1) with (r^0+r^1) twice (that is, $(r^0+r^1)^2$), we would get terms like $r^1b^1c^1d^1$ twice. This would mean that the apples were being considered as distinct. If we include only (r^0+r^1) once, we would never get terms such as $r^2b^0c^0d^1$, so we could never count the selection of both apples.

So what must we do? We need a way to adjust the apple enumerator so that it reflects the possibility of selecting either 0, 1, or 2 apples, and each of these in only one way. If, as in our first attempt, we replace (a^0+a^1) with $(r^0+r^1)^2 = (r^0+2r^1+r^2)$, we can quickly see that it counts the selection of just a single apple twice. So then it seems that what we really want to accomplish will be done by replacing (a^0+a^1) by $(r^0+r^1+r^2)$.

Exercise 10.4. Consider a fruit basket containing two red apples, one banana, one cantaloupe, and one date.

(i) Expand $(r^0+r^1+r^2)(b^0+b^1)(c^0+c^1)(d^0+d^1)$ and verify directly that all possible ways of selecting some fruit from the basket are represented correctly.

(ii) Replace all variables by x in the above expression to get $(1+x+x^2)(1+x)^3$. Then rewrite this in the form $(1+x)^4+x^2(1+x)^3$ and, using the binomial theorem, show that:

$$\begin{aligned}(1+x+x^2)(1+x)^3 &= \sum_{i=0}^{5}\left[\binom{4}{i}+\binom{3}{i-2}\right]x^i \\ &= 1+4x+7x^2+7x^3+4x^4+x^5,\end{aligned}$$

where $\binom{n}{i}$ is understood to be 0 whenever i is outside the interval $[0, n]$.

(iii) For each $i = 0$ to 5, explain by a direct counting argument that the coefficient of x^i is indeed the number of ways that one may select i pieces of fruit from this basket.

Exercise 10.5. Consider a fruit basket containing three each of apples, bananas, cantaloupes, and dates.

(i) Find the enumerator for each kind of fruit, replace all variables by x and write down the generating function (as a product) for selecting i pieces of fruit from this basket. Let $g(x)$ denote this generating function.

(ii) Factor $(1+x+x^2+x^3)$ and rewrite $g(x)$ in terms of powers of the factors. Then expand each of these powers using the binomial theorem.

(iii) Show that:

$$g(x) = \left[\sum_{h=0}^{4} \binom{4}{h} x^h\right] \left[\sum_{j=0}^{4} \binom{4}{j} x^{2j}\right]$$
$$= \sum_{i=0}^{12} \left[\sum_{j=0}^{4} \binom{4}{i-2j} \binom{4}{j}\right] x^i.$$

(iv) Interpret j in the above formula as the number of kinds of fruits in a selection which appear more than once. Show by directly listing all possibilities that $\left[\sum_{j=0}^{4} \binom{4}{3-2j}\binom{4}{j}\right]$ does count all of the ways that one may select 3 pieces of fruit from the basket.

(v) Show by listing all possibilities that $\left[\sum_{j=0}^{4} \binom{4}{7-2j}\binom{4}{j}\right]$ does count all of the ways of selecting 7 pieces of fruit from the basket.

EXERCISE 10.6. Consider a fruit basket containing four each of apples, bananas, cantaloupes, and dates. Give the enumerator (as a product) for each of the following counting problems. After you have set up each problem, go back and simplify each of the generating functions. Finally, try to compute the coefficients for the terms in the expansion of each generating function.

(i) In selecting fruit from the basket, you must select at least one of each kind of fruit.
(ii) In selecting fruit from the basket, you must select an odd number of each kind of fruit.
(iii) In selecting fruit from the basket, you must select an even number of each kind of fruit.

4. The Great Enumerator

Consider our fruit basket again and suppose that it contains 5 apples. Then the apple enumerator is $(a^0 + a^1 + a^2 + a^3 + a^4 + a^5)$. In general, if the basket contains m apples, the apple enumerator will be $(a^0 + a^1 + \cdots + a^m)$. What if we wish to consider a masket with an unlimited supply of apples? In this case, the natural choice for the enumerator is the infinite polynomial

$$(a^0 + a^1 + \cdots + a^m + \cdots).$$

This leads naturally to a question of whether we can do algebra with infinite polynomials. The answer, of course, is yes; operations of addition and multiplication are defined exactly as in the finite case. For example, consider our basket with an unlimited supply of apples, and exactly one banana. The generating function for selecting fruit from the basket is either

$$(a^0 + a^1 + \cdots + a^m + \cdots)(b^0 + b^1)$$

or

$$(1 + x + \cdots + x^m + \cdots)(1 + x),$$

depending upon whether we wish the terms to represent the actual choices for selecting i pieces of fruit, or simply the number of such selections.

Let us first consider $(a^0 + a^1 + \cdots + a^m + \cdots)(b^0 + b^1)$. Term-by-term multiplication gives us:

$$a^0b^0 + a^0b^1 + a^1b^0 + a^1b^1 + a^2b^0 + a^2b^1 + \cdots + a^mb^0 + a^mb^1 + \cdots .$$

Turning instead to $g(x) = (1 + x + \cdots + x^m + \cdots)(1 + x)$, we easily see that

$$\begin{aligned} g(x) &= (1 + x + \cdots + x^m + \cdots) + (x + x^2 + \cdots + x^m + \cdots) \\ &= 1 + 2x + 2x^2 + 2x^3 + \cdots + 2x^m + \cdots . \end{aligned}$$

We relate these two expansions by observing that, for all $m > 0$, $a^{m-1}b^1$ and a^mb^0 represent the two ways one may select m pieces of fruit from this basket.

Now, let's suppose that the basket contains an unlimited supply of both apples and bananas. Again, we have infinite polynomials which model this case:

$$(a^0 + a^1 + \cdots + a^m + \cdots)(b^0 + b^1 + \cdots + b^m + \cdots)$$

or

$$(1 + x + x^2 + \cdots + x^m + \cdots)^2.$$

Expanding the first product and grouping by the sum of the exponents, we get:

$$\begin{aligned} a^0b^0 + \big(a^0b^1 + a^1b^0\big) + \big(a^0b^2 + a^1b^1 + a^2b^0\big) + \cdots \\ + \big(a^0b^m + a^1b^{m-1} + \cdots + a^mb^0\big) + \cdots . \end{aligned}$$

Expanding $(1 + x + x^2 + \cdots + x^m + \cdots)^2$, we get:

$$1 + 2x + 3x^2 + \cdots + (m + 1)x^m + \cdots .$$

Thus, for example, we may select 5 pieces of fruit from our basket in 6 different ways. Using the first model, we identify the six ways as: 5 apples, 0 bananas; 4 apples, 1 banana; 3 apples, 2 bananas; 2 apples, 3 bananas; 1 apple, 4 bananas; or 0 apples, and 5 bananas.

It is clear that, if we wish to pursue the investigation of unrestricted selections, we will need to be very comfortable with the multiplication of infinite polynomials. In particular, we will need to be comfortable with products involving the enumerator $(1 + x + x^2 + \cdots + x^m + \cdots)$. This particular enumerator is so useful, we will call it the *great enumerator*. One product involving the great enumerator which is of particular interest is

$$(1 - x)(1 + x + x^2 + \cdots + x^m + \cdots) = 1.$$

Thus, surprisingly, we may identify the great enumerator by the notation

$$\frac{1}{1 - x} = (1 + x + x^2 + \cdots + x^m + \cdots).$$

5. Algebra Counts! – Exercise Set 2

EXERCISE 10.7. Let $(a_0 + a_1x + a_2x^2 + \cdots + a_mx^m + \cdots)$ denote an arbitrary infinite polynomial and consider the product

$$\frac{1}{1 - x}(a_0 + a_1x + a_2x^2 + \cdots + a_mx^m + \cdots).$$

(i) What is the coefficient of x_n in the expansion of this product?
(ii) Use your preceding answer to compute the coefficient of x_n in $\frac{1}{(1-x)^2}$.
(iii) Use your two preceding answers to compute the coefficient of x_n in $\frac{1}{(1-x)^3}$.

EXERCISE 10.8. Consider Pascal's Triangle:

$$\begin{array}{ccccccccccc} &&&&& 1 &&&&& \\ &&&& 1 && 1 &&&& \\ &&& 1 && 2 && 1 &&& \\ && 1 && 3 && 3 && 1 && \\ & 1 && 4 && 6 && 4 && 1 & \\ 1 && 5 && 10 && 10 && 5 && 1 \\ &&&&& \vdots &&&&& \\ 1 & n & \binom{n}{2} &&& \cdots &&& \binom{n}{2} & n & 1 \end{array}$$

(i) Recall that the coefficients of the expansion of $(1+x)^n$ form the n^{th} row of Pascal's Triangle. Observe that, for $n = 1, 2, 3$, the coefficients of the expansion of $\frac{1}{(1-x)^n}$ form the n^{th} diagonal of Pascal's Triangle.

(ii) Using the binomial identities, give a proof by induction for:

$$\frac{1}{(1-x)^n} = 1 + \binom{n}{1}x^1 + \binom{n+1}{2}x^2 + \cdots + \binom{n+i-1}{i}x^i + \cdots.$$

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