LISTING THE POSITIVE RATIONALS

JACK E. GRAVER

Abstract. The rational numbers are countable. The usual proof demonstrates that there exists a one-to-one function from the natural numbers onto the positive rational numbers or simply a list of all positive rationals (without repeats). But the list is not explicitly given. That is, there is no easy way to say which rational is 150th in the list or where $\frac{21}{13}$ appears in the list. In this paper, we discuss a different listing of the positive rationals for which we can easily answer such questions.

Start with a quick review of the "traditional" list. Consider the infinite array with the rows and columns indexed by the positive integers. The entry in the jth row and ith column is $\frac{i}{j}$. The list is then formed by moving along the diagonals from lower left to upper right including only the rational numbers that are reduced (numerator and denominator have no common factor other than 1). The first thirty entries in this list are included.

	1	2	3	4	5	6	7	8	9	...
1	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{6}{1}$	$\frac{7}{1}$	$\frac{8}{1}$	$\frac{9}{1}$	...
2	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	$\frac{6}{2}$	$\frac{7}{2}$	$\frac{8}{2}$	$\frac{9}{2}$	...
3	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$	$\frac{7}{3}$	$\frac{8}{3}$	$\frac{9}{3}$	...
4	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$	$\frac{7}{4}$	$\frac{8}{4}$	$\frac{9}{4}$	...
5	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	$\frac{6}{5}$	$\frac{7}{5}$	$\frac{8}{5}$	$\frac{9}{5}$	...
6	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$\frac{7}{6}$	$\frac{8}{6}$	$\frac{9}{6}$	...
7	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{6}{7}$	$\frac{7}{7}$	$\frac{8}{7}$	$\frac{9}{7}$	...
8	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	$\frac{8}{8}$	$\frac{9}{8}$	...
9	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{6}{9}$	$\frac{7}{9}$	$\frac{8}{9}$	$\frac{9}{9}$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

1	$\frac{1}{1}$	11	$\frac{5}{1}$	21	$\frac{7}{1}$
2	$\frac{1}{2}$	12	$\frac{1}{6}$	22	$\frac{1}{8}$
3	$\frac{2}{1}$	13	$\frac{2}{5}$	23	$\frac{2}{7}$
4	$\frac{1}{3}$	14	$\frac{3}{4}$	24	$\frac{4}{5}$
5	$\frac{3}{1}$	15	$\frac{4}{3}$	25	$\frac{5}{4}$
6	$\frac{1}{4}$	16	$\frac{5}{2}$	26	$\frac{7}{2}$
7	$\frac{2}{3}$	17	$\frac{6}{1}$	27	$\frac{8}{1}$
8	$\frac{3}{2}$	18	$\frac{1}{7}$	28	$\frac{1}{9}$
9	$\frac{4}{1}$	19	$\frac{3}{5}$	29	$\frac{3}{7}$
10	$\frac{1}{5}$	20	$\frac{5}{3}$	30	$\frac{7}{3}$

To actually generate the list we need to describe the algorithm for moving from one entry to the next. Then the only way to answer the question "what rational number is 150th in the list?" we must construct the first 150 rationals in the list. To answer the question "where is $\frac{21}{13}$ in the list?" we must construct the list until $\frac{21}{13}$ shows up.
The algorithm for constructing the list is:

Start with
0) Let $\ell(1) = \frac{1}{1}$.
1) Given $\ell(i) = \frac{p}{q}$
if $q = 1$, then let $i = i + 1$, $p = 1$, $q = p + 1$ and return to 1)
else, let $r = p + 1$ and $s = q - 1$;
2) if $\frac{r}{s}$ is reduced, $i = i + 1$, $p = r$, $q = s$ and return to 1);
else, let $r = r + 1$ and $s = s - 1$ and repeat 2).

This is a very difficult algorithm to use. In particular, checking whether $\frac{r}{s}$ is reduced or not can be very time consuming. The rest of this paper is devoted to developing a much better algorithm:

$$\ell(1) = \frac{1}{1} \text{ and, for all } i, \ell(i+1) = \frac{1}{\lfloor \ell(i) \rfloor + 1 - \{\ell(i)\}},$$

where $\lfloor x \rfloor$ is the floor or integer part function and $\{x\}$ is the fractional part function.

Theorem 1. *The list given by $\ell(n)$, for $n = 1, 2, \ldots,$ is a complete listing of all positive rational numbers without repeats.*

The key to proving this theorem is the construction of the *rational tree.*

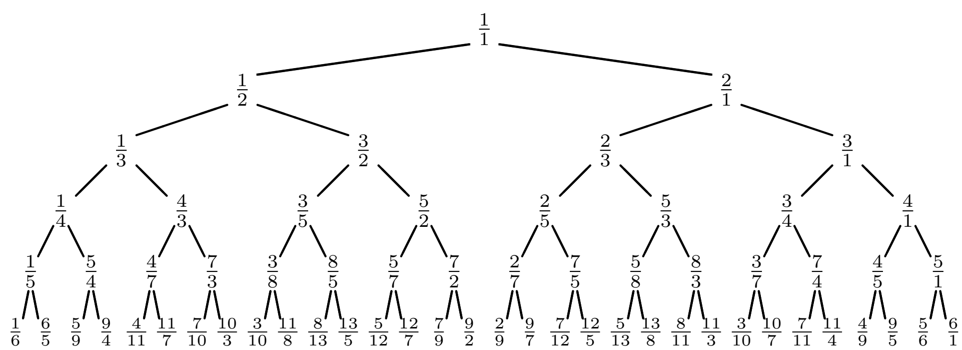

FIGURE 1. The rational tree T_Q

The rational tree T_Q is an infinite, binary tree with vertices consisting of the positive rational numbers. It is defined inductively as follows: the root is $\frac{1}{1}$; the left child of $\frac{p}{q}$ is $\frac{p}{p+q}$ and the right child of $\frac{p}{q}$ is $\frac{p+q}{q}$. The first six generations are pictured above. A version of this tree first appeared in 1858 in a paper by M. A. Stern [4] and is often referred to as the Stern-Brocot Tree. The following two lemmas tell us that each positive rational number appears exactly once in in the tree and that algorithm corresponds to moving through the tree row at a time. Together they prove the theorem.

Lemma 1. *Each positive rational number appears exactly once in T_Q.*

Proof. We proceed by induction on $p+q$. We start by observing that $\frac{1}{1}$ does appear in the tree as the root and indeed can only appear as the root. Our induction hypothesis is that every rational number $\frac{p}{q}$ (represented in lowest terms) with $p+q<n$ appears in the tree exactly once. Let $\frac{r}{s}$ be any rational with $r+s=n$. If $r>s$ then $\frac{r}{s}$ can only be the right hand child of $\frac{r-s}{s}$; since $(r-s)+s=r<n$, $\frac{r-s}{s}$ appears in the tree exactly once and therefore $\frac{r}{s}$ appears in the tree exactly once. Similarly, If $r<s$ then $\frac{r}{s}$ can only be the left hand child of $\frac{r}{s-r}$; since $r+(s-r)=s<n$, $\frac{r}{s-r}$ appears in the tree exactly once and therefore $\frac{r}{s}$ appears in the tree exactly once. $\square$

Lemma 2. *$\ell(i+1)$ appears to the immediate right of $\ell(i)$ in T_Q unless $\ell(i)$ is the right most entry in a row; in that case, $\ell(i+1)$ is the left most entry in the next row.*

Proof. There are two cases to consider. First assume that $\ell(i)$ is the right most entry in a row. Note that the j^{th} row starts with $\frac{1}{j}$ and ends with $\frac{j}{1}$ and assume that $\ell(i)=\frac{j}{1}$. By direct computation $\ell(i+1)=\frac{1}{j+1-0}$ which is the first entry in the next row.

Now suppose that $\ell(i)$ is the not right most entry in a row and let $\frac{p}{q}$ be the first common ancestor of $x=\ell(i)$ and y the vertex to its immediate right. Then $x=\frac{p+m(p+q)}{p+q}$ while $y=\frac{p+q}{q+m(p+q)}$. Hence $\lfloor x\rfloor=m$ and $\{x\}=\frac{p}{p+q}$ and $\ell(i+1)=\frac{1}{m+1-\frac{p}{p+q}}=y$. $\square$

This new list ℓ is much easier to compute inductively. However, computing $\ell(150)$ still requires 150 iterations and computing the position of $\frac{21}{13}$, an unknown number of iterations. Can we find a short-cut? The answer is yes.

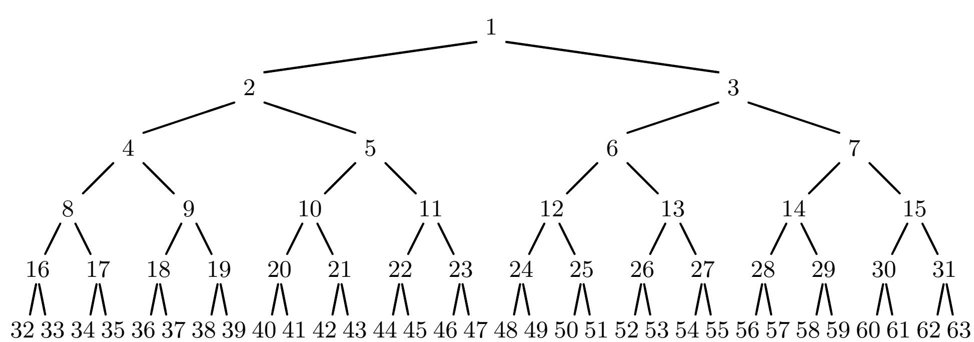

FIGURE 2. The counting number tree T_N

Consider the counting number tree T_N: the vertices are the positive integers in the position obtained by simply counting the nodes row at a time from left to right. Then $\ell(i)$ is the rational number occupying the position in T_Q corresponding to the position of i in T_N.

Taking a closer look at the structure of T_N we note that the left child of the vertex i is simply $2i$ while the right child is $2i+1$. Hence the binary representation of the left child of the vertex i is obtained from the binary representation of i by adding a 0 at its right end and binary representation of the right child of the vertex i is obtained from the binary representation of i by adding a 1 at its right end. This is also obvious in the tree T_B which is simply T_N with its vertices written in the binary number system.

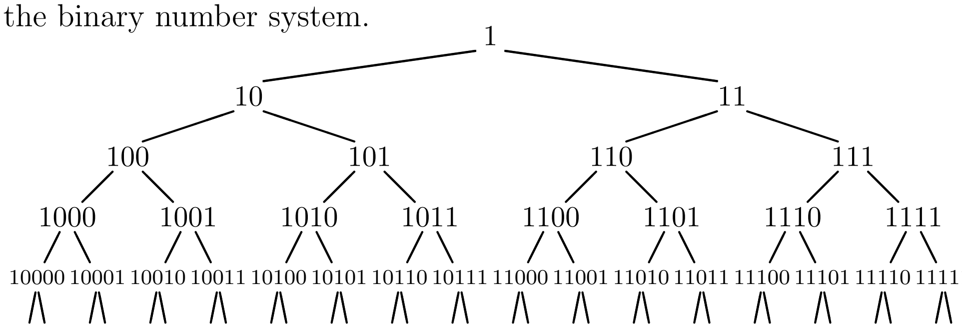

FIGURE 3. The binary counting tree T_B

One may think of the binary representation of i as the "MapQuest" directions to its position. For example, 50 is 110010 when written in binary. Starting at 1, the one in the second position tells us to double 1 and add 1 yielding 3. The next two zeros tell us to double 3 twice to get 6 and then 12. The last 1 and 0 instruct us to double 12 and add 1 to get 25 and then doubling again to get 50. Applying these same directions to T_Q we can compute $\ell(50)$: starting at $\frac{1}{1}$ the second 1 directs us to the right child $\frac{2}{1}$; taking the left child twice gives $\frac{2}{3}$ and $\frac{2}{5}$; following this with a right and left gives $\frac{7}{5}$ and $\frac{7}{12}$. Hence $\ell(50) = \frac{7}{12}$.

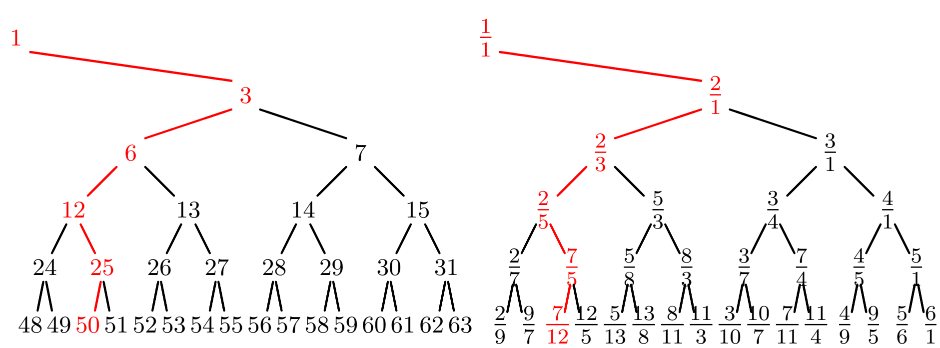

FIGURE 4. Computing $\ell(50)$

We can actually give a formula for $\ell(n)$. The *binary cipher* for n can be described in terms of the binary representation of n: the sequence $a_0, a_1, a_2, \ldots, a_k$ where the binary representation consists of 1 followed by a_0 0s, then a_1 1s, then a_2 0s, etc.. In this sequence of integers a_0 is non-negative the rest are positive. In constructing $\ell(n)$ we move from $\frac{1}{1}$ to $\frac{1}{1+a_0}$, then to $\frac{1+(1+a_0)a_1}{1+a_0} = \frac{1+a_1+a_0a_1}{1+a_0}$ (or directly from $\frac{1}{1}$ to $\frac{1+a_1+a_0a_1}{1+a_0} = \frac{1+a_1}{1}$ when $a_0 = 0$), then to $\frac{1+a_1+a_0a_1}{1+a_0+(1+a_1+a_0a_1)a_2} = \frac{1+a_1+a_0a_1}{1+a_0+a_2+a_1a_2+a_0a_1a_2}$ and so on.

Let $a_0, a_1, a_2, \ldots, a_k$ be the binary cipher of some integer. We define N_k to be 1 plus the sum of all products of the a_i with increasing indices alternating in parity and ending with an odd index and we define D_k to be 1 plus the sum of all products of the a_i with increasing indices alternating in parity and ending with an even index.

Lemma 3. *Let $a_0, a_1, a_2, \ldots, a_k$ be the binary cipher of some integer. Then:*

$$N_k = \begin{cases} N_{k-1}, & k \text{ even} \\ N_{k-1} + D_{k-1}a_k, & k \text{ odd} \end{cases} \quad \& \ D_k = \begin{cases} D_{k-1}, & k \text{ odd} \\ D_{k-1} + N_{k-1}a_k, & k \text{ even} \end{cases}$$

Proof. Suppose that k is even. Then all products of the a_i with increasing indices alternating in parity and ending with an odd index are included in N_{k-1} and so $N_k = N_{k-1}$. Now assume that k is odd and and partition the terms of N_k in to those that end in a_k and those that do not. The sum of those that do not end in a_k is simply N_{k-1} while the sum of those terms that do end in a_k is $D_{k-1}a_k$. Hence the formula for N_k holds. The verification of the formula for D_k is similar. □

Theorem 2. *Let n be the positive integer with binary cipher $a_0, a_1, a_2 \ldots a_k$. Then $\ell(n) = \frac{N_k}{D_k}$, where N_k is 1 plus the sum of all products of the a_i with increasing indices alternating in parity and ending with an odd index while D_k is 1 plus the sum of all products of the a_i with increasing indices alternating in parity and ending with an even index.*

Proof. Suppose that the binary cipher of n is simply a_0. Then the binary representation of n is 1 followed by a_0 0s. But then $n = 2^{a_0}$ and $\ell(n) = \frac{1}{1+a_0} = \frac{N_0}{D_0}$. We proceed by induction on the index k. Suppose that the binary cipher of n is $a_0, a_1, \ldots, a_k$ and that $\ell(m) = \frac{N_h}{D_h}$ for all n with shorter binary cphers. In particular, $\ell(m) = \frac{N_{k-1}}{D_{k-1}}$ where m has binary cipher $a_0, a_1, \ldots, a_{k-1}$. If k is even, then the binary representation of n is the binary representation of m followed by a_k 0s. Hence $\ell(n) = \frac{N_{k-1}}{D_{k-1}+N_{k-1}a_k} = \frac{N_k}{D_k}$. If k is odd, then the binary representation of n is the binary representation of m followed by a_k 1s. Hence $\ell(n) = \frac{N_{k-1}+D_{k-1}a_k}{D_{k-1}} = \frac{N_k}{D_k}$. □

Applying this result to 150, we have that the binary representation of 150 is 10010110 so the *binary cipher* of 150 is $a_0 = 2$, $a_1 = 1$, $a_2 = 1$, $a_3 = 2$ and $a_4 = 1$. So $\ell(150)$ equals

$$\frac{1 + a_1 + a_3 + a_0a_1 + a_0a_3 + a_2a_3 + a_1a_2a_3 + a_0a_1a_2a_3}{1 + a_0 + a_2 + a_4 + a_1a_2 + a_1a_4 + a_3a_4 + a_0a_1a_2 + a_0a_1a_4 + a_0a_3a_4 + a_2a_3a_4 + a_1a_2a_3a_4 + a_0a_1a_2a_3a_4}$$
$$= \frac{1+1+2+2+4+2+2+4}{1+2+1+1+1+1+1+2+2+2+4+2+2+4} = \frac{18}{25}$$

Finally, to answer the question "just where does $\frac{21}{13}$ fit in the list?" we work backwards to construct the binary representation of its position number: $\frac{21}{13}$ is an improper fraction so it is the right hand child of $\frac{8}{13}$ and the last binary digit of its position is 1; $\frac{8}{13}$ is proper so it is the left hand child of $\frac{8}{5}$ and the second last binary digit of the position of is $\frac{21}{13}$ 0. This procedure is summarized in the following table (moving from right to left)

1	0	1	0	1	0	1
$\frac{1}{1}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{5}$	$\frac{8}{5}$	$\frac{8}{13}$	$\frac{21}{13}$

. Since 1010101 is the binary representation of 85, $\ell^{-1}(\frac{21}{13}) = 85$. So we have a simple algorithm for computing the values for ℓ^{-1}. Is there a formula for ℓ^{-1}?

The function ℓ has several interesting properties to investigate. For example, note that $\ell(\lfloor \frac{2^8}{3} \rfloor) = \ell(85) = \frac{21}{13} = \frac{f_8}{f_7}$, where f_i denotes the ith Fibonacci number. This is no accident!

References

[1] M. Aigner and G. Ziegler (2004), *Sets, functions and the continuum hypothesis,* **Proofs from the Book,** Chapter 16.

[2] Calkin, Neil and Wilf (2000), *Recounting the rationals,* American Mathematical Monthly, Vol. 107, No. 4, 360-363.

[3] Newman, M. (2003), *Recounting the rationals revisited*, American Mathematical Monthly, Vol. 110 No. 7, 642-643.

[4] M. A. Stern (1858) *Über eine zahlentheoretische Funktion,* Journal für dir reine angewandte Mathematik Vol. 55, 193-220.