Latin Squares and Projective Planes

Combinatorics Seminar, SPRING, 2010

CHAPTER 1

Finite Fields

1. Examples

F_4 (x^2+x+1 is irreducable)

	x	$x+1$
x	$x+1$	1
$x+1$	1	x

$F_4 = \{(a,b) | a,b \in \mathbb{Z}_2\}$
$(a,b)+(c,d) = (a+c, b+d)$
$(a,b)(c,d) = (ac+ad+bc, ac+bd)$

F_8 (x^3+x+1 is irreducable)

	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
x	x^2	x^2+x	$x+1$	1	x^2+x+1	x^2+1
$x+1$	x^2+x	x^2+1	x^2+x+1	x^2	1	x
x^2	$x+1$	x^2+x+1	x^2+x	x	x^2+1	1
x^2+1	1	x^2	x	x^2+x+1	$x+1$	x^2+x
x^2+x	x^2+x+1	1	x^2+1	$x+1$	x	x^2
x^2+x	x^2+1	x	1	x^2+x	x^2	$x+1$

F_9 (x^2+1 is irreducable)

	2	x	$2x$	$x+1$	$x+2$	$2x+1$	$2x+2$
2	1	$2x$	x	$2x+2$	$2x+1$	$x+2$	$x+1$
x	$2x$	2	1	$x+2$	$2x+2$	$x+1$	$2x+1$
$2x$	x	1	2	$2x+1$	$x+1$	$2x+2$	$x+2$
$x+1$	$2x+2$	$x+2$	$2x+1$	$2x$	1	2	x
$x+2$	$2x+1$	$2x+2$	$x+1$	1	x	$2x$	2
$2x+1$	$x+2$	$x+1$	$2x+2$	2	$2x$	x	1
$2x+2$	$x+1$	$2x+1$	$x+2$	x	2	1	$2x$

CHAPTER 2

Combinatorics

1. Introduction

2. Latin Squares

3. Projective Planes

DEFINITION 2.3.0. *By a set system Σ we mean a pair of disjoint sets (X, Y) such that the elements of Y are identified with subsets of X:*

(i) *we abuse notation and write $y \in \mathcal{P}(X)$ and $x \in y$;*
(ii) *we abuse notation and write $x \in \mathcal{P}(Y)$, identifying x with $\{y | x \in y\}$;*
(iii) *if all distinct pairs $x, x' \in X$ are distinct as subsets in Y, $\Sigma^* = (Y, X)$ is also a set system, called the dual to $\Sigma = (X, Y)$;*
(iv) *Σ^{**} has a natural identification with Σ.*

DEFINITION 2.3.1. *A set system $\Pi = (P, L)$ (points, lines) is a projective plane if*

(i) *for distinct points $p, p' \in P$, there exists a unique line $\ell \in L$ such that $p, p' \in \ell$;*
(ii) *for distinct lines $\ell, \ell' \in L$, $|\ell \cap \ell'| = 1$;*
(iii) *there exist 4 distinct point no three of which are collinear.*

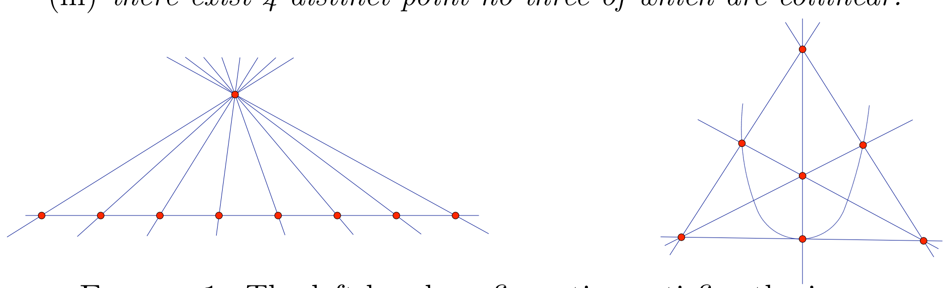


FIGURE 1. The left-hand configuration satisfies the incidence axioms but not axiom (iii) (non-degeneracy); the right-hand configuration is the smallest projective plane.

LEMMA 2.3.2a. *Let $\Pi = (P, L)$ be a projective plane.*

(i) *There exist four lines no three of which are concurrent.*

(ii) $\Pi^* = (L, P)$ *is also a projective plane.*
(iii) *If* $\ell, \ell' \in L$ *then there exists a point* p *such that* $p \notin \ell \cup \ell'$.
(iv) *If* $p, p' \in P$ *then there exists a line* ℓ *such that* $p, p' \notin \ell$.

PROOF. (i) Let $p_1, p_2, p_3, p_4 = p_0$ be four point no three of which are collinear and let ℓ_i be the line through p_{i-1} and p_i. WLOG suppose that ℓ_1, ℓ_2 and ℓ_3 were concurrent at p. Since $\ell_1 \cap \ell_2 = p_1$, $p = p_1$; since $\ell_2 \cap \ell_3 = x_2$, $p = p_2$. But $p_1 \neq p_2$, contradiction.
(ii) The incidence axioms are symmetric and (i) is the non-degeneracy axiom for the dual.
(iii) Suppose $P = \ell \cup \ell'$. By the non-degeneracy axiom, we may choose $p \neq q \in \ell - \ell \cap \ell'$ and $p' \neq q' \in \ell' - \ell \cap \ell'$. Let ℓ_p denote the line through p and p' and ℓ_q denote the line through q and q'. But then $\ell_p \cap \ell_q$ is a point not on $\ell \cup \ell'$ - contradiction.
(iv) Follows from (iii) by duality.

□

THEOREM 2.3.2. *If* $\Pi = (P, L)$ *is a finite projective plane there exists an integer* $n > 1$ *such that:*
(i) *each line contains exactly* $n + 1$ *points;*
(ii) *each point is contained in exactly* $n + 1$ *lines;*
(iii) *there are exactly* $n^2 + n + 1$ *points;*
(iv) *there are exactly* $n^2 + n + 1$ *lines.*

PROOF. (i) and (ii). Let ℓ be any line and p any point not on ℓ. Then each line containing p intersects ℓ in a different point and each point on ℓ lies on a different line containing p. Hence $|\ell| = |p|$. It follows from this and Lemma 2.3.2a (iii) & (iv) that all lines and all sets of lines through a common point have the same cardinality; $n+1$. We note that the non-degeneracy condition forces a line and hence all lines to have at least 3 points.

(iii) and (iv) Finally let p be any point and let $\ell_1, \ldots, \ell_{n+1}$ be the lines through p. Clearly $P = \cup_{i=1}^{n+1} \ell_i$ and $|P| = 1 + (n+1)n = n^2 + n + 1$; $|L| = n^2 + n + 1$ by duality. □

DEFINITION 2.3.3. *A set system* $\Theta = (P, L)$ *(points, lines) is an affine plane if*
(i) *for distinct points* $p, p' \in P$, *there exists a unique line* $\ell \in L$ *such that* $p, p' \in \ell$;
(ii) *for each line* ℓ *and each point* p *not on* ℓ *there exists a unique line* ℓ', *such that* $p \in \ell'$ *and* $\ell \cap \ell' = \emptyset$ *(*ℓ *and* ℓ' *are parallel);*
(iii) *there exist three non-collinear points.*

THEOREM 2.3.4. *Let $\mathbb{K}$ be any field, let $P = \mathbb{K}^2$ and let $L = \{\{(x,y)|ax+by+c=0\}|a,b,c \in \mathbb{K}; a,b$ not both $0\}$. Then $\Theta = (P,L)$ is a affine plane.*

PROOF. It is convenient to write the equations for the lines in standard form. Let $\ell = \{(x,y)|ax+by+c=0\}$; if $b \neq 0$, $\ell = \{(x,y)|y = \frac{-a}{b}x + \frac{-c}{b}\}$ and, if $b = 0$, $\ell = \{(x,y)|x = \frac{-c}{a}\}$.

Axiom (i). Let $p_1 = (x_1,y_1)$ and $p_2 = (x_2,y_2)$ be two distinct points. Case 1, $x_1 = x_2 = c_0$ then $p_1, p_2 \in \ell$ where $\ell = \{(x,y)|x = c_0\}$. Suppose that $p_1, p_2 \in \ell$ where $\ell = \{(x,y)|y = ax+c\}$. Then $y_1 = ax_1 + c$ and $y_2 = ax_2 + c$; subtracting these equations yields $y_1 = y_2$ and $p_1 = p_2$. Case 2, $x_1 \neq x_2$. It is clear that p_1 and p_2 could not lie on the same vertical line; but the system $y_1 = ax_1 + c$ and $y_2 = ax_2 + c$ has a unique solution: $a = \frac{y_1 - y_2}{x_1 - x_2}$ and $c = y_1 - \frac{y_1 - y_2}{x_1 - x_2}x_1$.

Axiom (ii) If $p_1 = (x_1,y_1)$ is not on ℓ_1 given by $x = c$ then $x = x_1$ is the unique line through p_1 and parallel to ℓ_1; if $p_1 = (x_1,y_1)$ is not on ℓ_1 given by $y = ax+c$ then $y = ax + (y_1 - ax_1)$ is the unique line through p_1 and parallel to ℓ_1.

Axiom (iii) No three of the points (0,0), (1,0), (0,1) and (1,1) collinear. □

LEMMA 2.3.5a. *Let $\Theta = (P,L)$ an affine plane. The relation $\parallel$ on L, where $\ell \parallel \ell'$ whenever $\ell \cap \ell' = \emptyset$ or $\ell = \ell'$, is an equivalence relation.*

PROOF. It is reflexive and symmetric by definition. Assume that $\ell_1 \| \ell_2 \| \ell_3$. Suppose that $p \in \ell_1 \cap \ell_3$. Then $p \notin \ell_2$ but there can be only one line through p parallel to ℓ_2. Hence $\|$ is transitive. □

THEOREM 2.3.5. (i) *Let $\Pi = (P,L)$ be a projective plane and let $\ell_\infty \in L$. Then $\Theta = (P - \ell_\infty, L - \{\ell_\infty\})$ is an affine plane.*

(ii) *Let $\Theta = (P,L)$ an affine plane and let $E = \{e|e$ is an equivalence class of parallel lines$\}$. Let $P^+ = P \cup E$; for $\ell \in L$ let $\ell^+ = \ell \cup e$ where $\ell \in e$; finally let $L^+ = \{\ell^+|\ell \in L\} \cup \{E\}$. Then $\Pi = (P^+, L^+)$ is a projective plane.*

COROLLARY 2.3.6. (i) *If $\Theta = (P,L)$ is a finite affine plane then there is an integer $n > 1$ such that each line contains exactly n points, each point is contained in exactly $n+1$ lines, $|P| = n^2$ and $|L| = n^2 + n$. [n is called the order of Θ.]*

(ii) *If q is a prime power then there exists a projective plane and an affine plane of order n.*

EXAMPLE 2.3.7. *The left-hand configuration in Figure 2 is the affine plane of order 2 over $\mathbb{Z}_2$; the right-hand configuration is the projective plane of order 2 obtained by adding a line of points at infinity.*

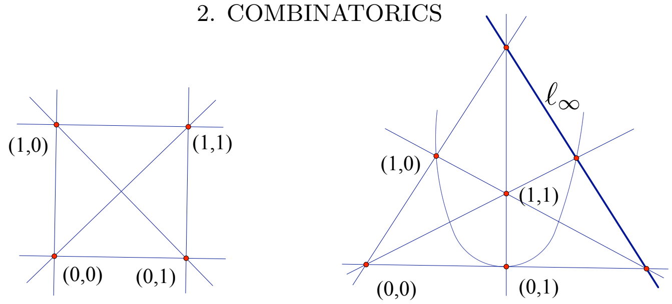


FIGURE 2

THEOREM 2.3.8. *(Bose 1938) There are $n-1$ mutually orthogonal Latin squares of order n if and only if there exists a projective plane of order n.*

PROOF. Assume that we have a complete set of orthogonal Latin squares of order n: $\mathbf{L}^1 \dots \mathbf{L}^{n-1}$. Define $\Pi = (P, L)$ as follows:

(i) $P = \{p_{ij} | 1 \leq i, j \leq n\} \cup \{p_0, p_1, \dots, p_{n-1}, p_\infty\}$;
(ii) $L = \{\ell^k_m | 1 \leq k \leq n-1; 1 \leq m \leq n\} \cup \{\ell^0_m, \ell^\infty_m\} \cup \{\ell_\infty\}$, where
  (a) $\ell^k_m = \{p_{ij} | \mathbf{L}^k_{ij} = m\} \cup \{p_k\}$, for $1 \leq k \leq n-1; 1 \leq m \leq n$,
  (b) $\ell^0_m = \{p_{mj} | 1 \leq j \leq n\} \cup \{p_0\}$,
  (c) $\ell^\infty_m = \{p_{jm} | 1 \leq j \leq n\} \cup \{p_\infty\}$,
  (d) $\ell_\infty = \{p_0, p_1, \dots, p_{n-1}, p_\infty\}$.

It follows at once that:

(i) there are $n^2 + n + 1$ points;
(ii) there are $n^2 + n + 1$ lines;
(iii) each line contains $n + 1$ points;
(iv) each point lies on $n + 1$ lines; concurrent.

Axiom (ii). Two distinct lines intersect in exactly one point:

- $\ell_\infty \cap \ell^k_m = \{p_k\}$ $(k \in \{0, 1, \dots, n-1, \infty\})$;
- $\ell^k_r \cap \ell^k_s = \{p_k\}$ $(k \in \{0, 1, \dots, n-1, \infty\})$;
- $\ell^0_i \cap \ell^\infty_j = \{p_{ij}\}$, $(1 \leq i, j \leq n)$;
- $\ell^0_i \cap \ell^k_m = \{p_{ij}\}$ where $\mathbf{L}^k_{ij} = m$ $(1 \leq i \leq n$ and $1 \leq k \leq n-1)$;
- $\ell^\infty_j \cap \ell^k_m = \{p_{ij}\}$ where $\mathbf{L}^k_{ij} = m$ $(1 \leq j \leq n,\ 1 \leq k \leq n-1)$;
- $\ell^h_r \cap \ell^k_s = \{p_{ij}\}$ where $\mathbf{L}^h_{ij} = r$ and $\mathbf{L}^k_{ij} = s$ that is ij are the coordinates of the pair (r,s) in the Greco-Latin square $[\mathbf{L}^h, \mathbf{L}^k]$ $(1 \leq j \leq n,\ 1 \leq k \leq n-1)$.

Axiom (i). Consider two distinct points p and q. Let $\overline{\ell}_1, \dots, \overline{\ell}_{n+1}$ denote the $n+1$ lines containing p with p deleted. By axiom (ii) they are disjoint. By our counting arguments they partition the $(n+1)n = n^2+n$ points of $P - \{p\}$. Hence q lies on exactly one of these.

Axiom (iii) No three of $p_{11}, p_{12}, p_{21}, p_{22}$ are collinear.

Conversely, let $\Pi = (P, L)$ be a finite projective plane of order n.

(i) Select any line and label it ℓ_∞
(ii) Label its points p_k, for $k \in \{0, 1, \ldots, n-1, \infty\}$
(iii) Label the lines through p_k, ℓ_m^k where $m = 1, \ldots, n$.
(iv) Let p_{ij}denote the point on ℓ_i^0 and ℓ_j^∞
(v) Define $\mathbf{L}^k$, for $k \in \{1, \ldots, n-1\}$, by $\mathbf{L}_{ij}^k = m$ where ℓ_m^k is the line through p_{ij} and p_k
(vi) The entry in column x_i and row x_m is j where L_{ij} is the line containing x_i and x_m.

Since ℓ_i^0 and ℓ_m^k intersect, m appears somewhere in the i*th* row of $\mathbf{L}^k$; since ℓ_j^∞ and ℓ_m^k intersect, m appears somewhere in the j*th* column of $\mathbf{L}^k$. Hence $\mathbf{L}^k$ is a Latin square. Now for any distinct $k, h \in \{1, \ldots, n-1\}$ and any $r, s \in \{1, \ldots, n\}$, the pair (r, s) appears in the ij position of the Greco-Latin square $[\mathbf{L}^h, \mathbf{L}^k]$, where p_{ij} is the point of intersection of ℓ_r^h and ℓ_s^k. □

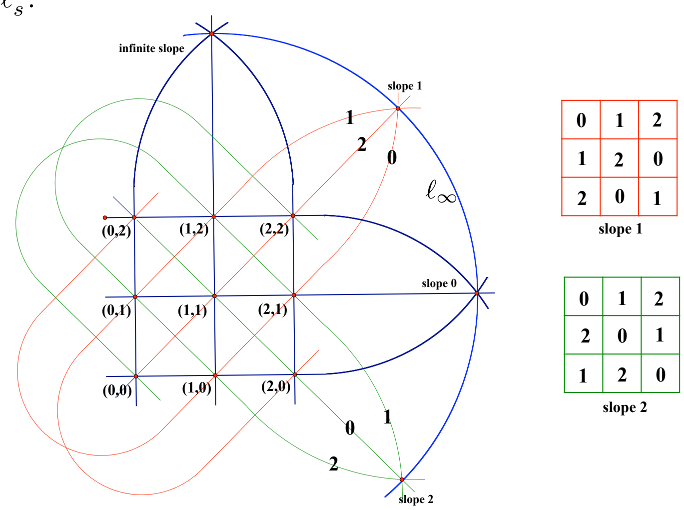


FIGURE 3

CONJECTURE 2.3.9. *There exists a finite projective plane (an affine plane, a complete set of orthogonal Latin squares) of order* n *if and only if* n *is a prime power. (This has been verified for* $n < 12$*.)*

DEFINITION 2.3.10. *Let* $\Pi = (P, L)$ *be a projective plane. By a triangle in* Π *we mean three non-collinear points* $V \subset P$ *(vertices) and the three lines containing the three pairs of these points* $S \subset L$ *(sides):* $V = \{p_1, p_2, p_3\}$, $S = \{\ell_1, \ell_2, \ell_3,$ *where* $p_i, p_j \in \ell_h$ $\{i, j, h\} = \{1, 2, 3, \}\}$.

THEOREM 2.3.11 (Desargues). *Let* $\Pi = (P, L)$ *be the projective plane over the field* $\mathbb{K}$*; let* $T = (V, S)$ *and* $T' = (V', S')$ *be two disjoint triangles of* Π*:* $V = \{p_1, p_2, p_3\}$, $S = \{\ell_1, \ell_2, \ell_3\}$, $V' = \{p_1', p_2', p_3'\}$, $S' = \{\ell_1', \ell_2', \ell_3'\}$. *Let* h_i *be the line that contains* p_i *and* p_i'*, for* $i = 1, 2, 3$

and let $q_i = \ell_i \cap \ell'_i$, *for* $i = 1, 2, 3$. *Then* h_1, h_2, h_3 *are concurrent if and only if* q_1, q_2, q_3 *are collinear.*

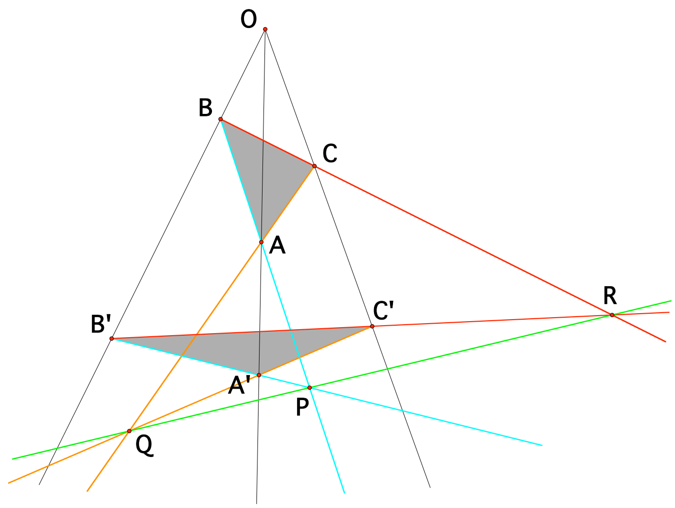


FIGURE 4

THEOREM 2.3.12 (Pappus). *Let* $\Pi = (P, L)$ *be the projective plane over the field* $\mathbb{K}$. *If the six vertices of a hexagon lie alternately on two lines, the three points of intersection of opposite sides are colinear.*

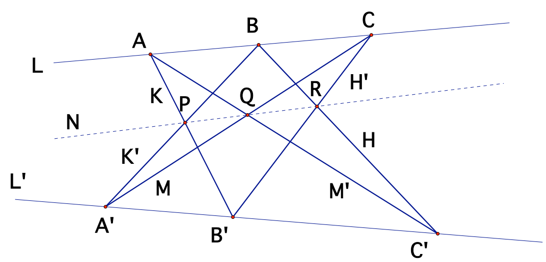


FIGURE 5

Some interesting geometric facts:

- A Projective plane is Desarguesian if and only if it may be constructed from a division ring.
- There exists a non-Desarguesian projective plane of order 9 and none with smaller order.
- A Projective plane may be constructed from a field if and only if it is Desarguesian and satisfies Pappus's Theorem.
- The algebraic theorem "All finite division rings are fields" implies Pappus's Theorem holds in all Desarguesian finite projective planes.
- No geometric proof is known!

NOTATION. Let $\Theta = (P, L)$ be an affine plane. We will denote the line through distinct points p and q by ℓ_{pq}.

DEFINITION 2.3.13. *Let $\Theta = (P, L)$ be an affine plane. A mapping $\sigma : P \to P$ is called a* <u>dilatation</u>*, if for any $p \neq q \in P$, $\sigma(q) \in \ell$, where ℓ is the line through $\sigma(p)$ parallel to ℓ_{pq}.*

EXAMPLE 2.3.14. *Let Θ be the affine plane $\mathbb{Z}_5 \times \mathbb{Z}_5$. Then τ, defined by $\tau(i, j) = (i + 1, j + 1)$, and δ, defined by $\delta(i, j) = (2i, 2j)$, are dilatations.*

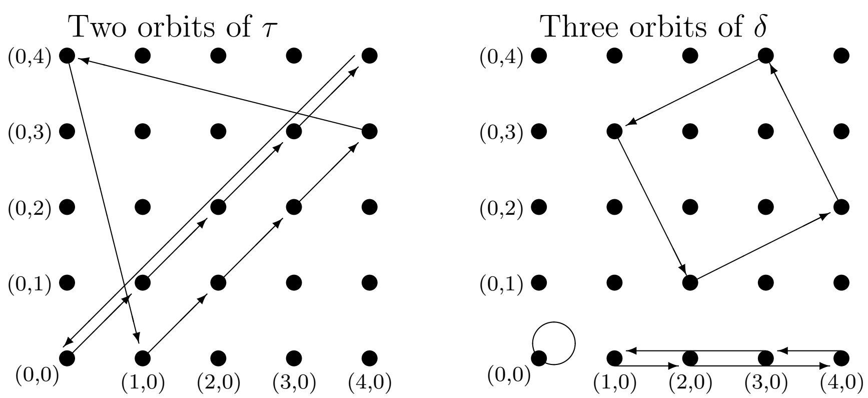


FIGURE 6

LEMMA 2.3.15a. *A dilatation σ on the affine plane $\Theta = (P, L)$ is uniquely determined by the image of any two distinct points. Specifically, let p and q distinct points: if r is any point not on ℓ_{pq}, $\sigma(r)$ is the unique point of intersection of the non-parallel lines ℓ'_{pr}, the line through $\sigma(p)$ parallel to ℓ_{pr}, and ℓ'_{qr}, the line through $\sigma(q)$ parallel to ℓ_{qr}. If s is any point on ℓ_{pq}, $\sigma(s)$ is the unique point of intersection of $\ell_{\sigma(p)\sigma(q)}$, the line through $\sigma(p)$ parallel to ℓ_{pq}, and ℓ'_{rs} the line through $\sigma(r)$ parallel to ℓ_{rs}, where r is any point not on ℓ_{pq}.*

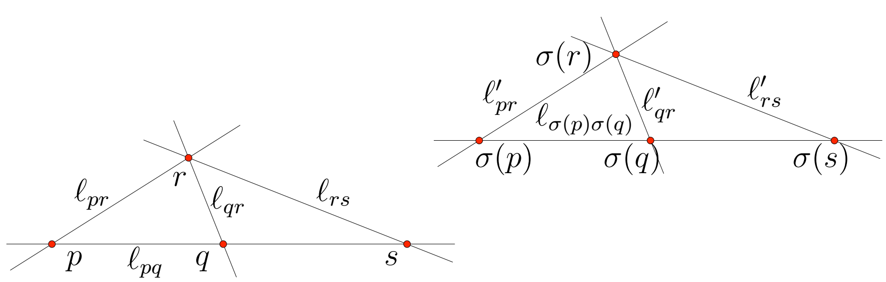


FIGURE 7

THEOREM 2.3.15. *Let $\Theta = (P, L)$ be an affine plane and $\sigma : P \to P$ a dilatation. Then either σ is one-to-one and onto or the image of σ is a single point, in which case we say that σ is degenerate.*

PROOF. Let p and q be distinct points and assume that $\sigma(p) = \sigma(q) = x$. Let r be any point not on ℓ, then $\sigma(r)$ is the unique point of intersection of ℓ'_{pr} through $x = \sigma(p)$ and ℓ'_{qr} through $x = \sigma(q)$. But, since $x = \sigma(p) = \sigma(q)$, this intersection is x. Hence every point not on ℓ_{pq} is mapped onto x. But, then every point not on ℓ_{pr} is also mapped onto x and therefore every point on ℓ_{pq} is also mapped onto x. We conclude that σ is degenerate or one-to-one.

Assume that σ is one-to-one. Let p and q be distinct points, let s be any other point on $\ell_{\sigma(p)\sigma(q)}$ and let r be any point not on $\ell_{\sigma(p)\sigma(q)}$. Let r' be the unique point of intersection of $\ell'_{\sigma(p)r}$, the line through p parallel to $\ell'_{\sigma(p)r}$, and $\ell'_{\sigma(q)r}$, the line through q parallel to $\ell'_{\sigma(q)r}$. It follows at once that $\sigma(r') = r$. Similarly, it is easy to see that s', the point of intersection of the line ℓ_{pq} and the line through r' parallel to ℓ_{rs}, is mapped onto s. Hence σ is onto. □

COROLLARY 2.3.16. *Let $\Theta = (P, L)$ be an affine plane and σ a non-degenerate dilatation. Then σ maps each line onto a parallel line. Specifically, σ maps ℓ_{pq} onto $\ell_{\sigma(p)\sigma(q)}$ for all distinct points p and q.*

DEFINITION 2.3.17. *Let $\Theta = (P, L)$ be an affine plane and σ a non-degenerate dilatation and p any point in P. Then any line containing p and $\sigma(p)$ is called a trace of σ.*

EXAMPLE 2.3.18. *Referring to Example 2.3.14, the traces of τ consists of the five lines of slope 1 and the traces of δ consist of the six lines through* $(0, 0)$.

LEMMA 2.3.20a. *Let $\Theta = (P, L)$ be an affine plane, let σ be a non-degenerate dilatation and let p and q be points that are not fixed by σ. Then if q is on the trace $\ell_{p\sigma(p)}$ so is $\sigma(q)$ and the trace $\ell_{q\sigma(q)} = \ell_{p\sigma(p)}$.*

THEOREM 2.3.20. *Let $\Theta = (P, L)$ be an affine plane and σ a non-degenerate dilatation. Then either*

(i) *σ is the identity map and all lines are traces or*
(ii) *σ admits exactly one fixed point and the traces of σ are the lines through the fixed point or*
(iii) *σ admits no fixed points and the traces of σ are the lines in a pencil of parallel lines.*

PROOF. (i) If σ fixes two points, it agrees with the identity at two points and must equal the identity map. In that case each line contains a point and its image.

(ii) Assume that p is the only fixed point of σ. Let q be any other point. Since ℓ_{pq} and $\ell_{\sigma(p)\sigma(q)} = \ell_{p\sigma(q)}$ are parallel and pass through p, they are equal. Hence ℓ_{pq} is the trace given by q.

(iii) Finally assume that σ has no fixed points. Let p be any point and consider the trace $\ell = \ell_{p\sigma(p)}$. Let q be any other point and consider the trace $\ell' = \ell_{q\sigma(q)}$. If $r \in \ell \cap \ell'$, ℓ and ℓ' both equal $\ell_{r\sigma(r)}$; otherwise ℓ and ℓ' are disjoint. Hence all traces are parallel to ℓ and, since every point lies on some trace, the collection of the traces of σ is the pencil of lines parallel to ℓ.

□

DEFINITION 2.3.21. *Let $\Theta = (P, L)$ be an affine plane. A non-degenerate dilatation δ is a dilation if it admits exactly one fixed point. A non-degenerate dilatation τ is a translation if it is the identity map or has no fixed points. If τ has no fixed points, the pencil of traces of τ is called the direction of τ.*

EXAMPLE 2.3.22. *The affine plane $\mathbb{Z}_5 \times \mathbb{Z}_5$ admits 100 dilatations:*

(i) *The 75 dilations δ_{ijm}, $i, j \in \mathbb{Z}_5$, $m = 2, 3, 4$ where $\delta_{ijm}(h, k) = (m(h-i)+i, m(k-j)+j)$. Geometrically δ_{ijm} fixes (i, j) and permutes the remaining four points on each line through (i, j). Specifically, the δ in Example 2.3.14 is δ_{002}.*

(ii) *The 25 translations τ_{ij}, $i, j \in \mathbb{Z}_5$, defined by $\tau_{ij}[(h, k)] = (h+i, k+j)$. τ_{00} is the identity. The direction of τ_{0k}, $k = 1, 2, 3, 4$, is the pencil of vertical lines; the direction of τ_{hk}, $h = 1, 2, 3, 4$, $k = 0, 1, 2, 3, 4$, is the pencil of lines of slope $\frac{k}{h}$. Geometrically, the translations cyclically permute the five points on each trace line. Specifically, the τ in Example 2.3.14 is τ_{11}.*

EXAMPLE 2.3.25. *Consider the field $\mathbb{F}_4$ and the affine space $\mathbb{F}_4 \times \mathbb{F}_4$. Let $\mathbb{F}_4 = \{0, 1, a, b\}$, where a and b are the roots of $x^2 + x + 1$.*

+	1	a	b
1	0	b	a
a	b	0	1
b	a	1	0

×	a	b
a	b	1
b	1	a

The 20 lines fall into 5 parallelism classes.

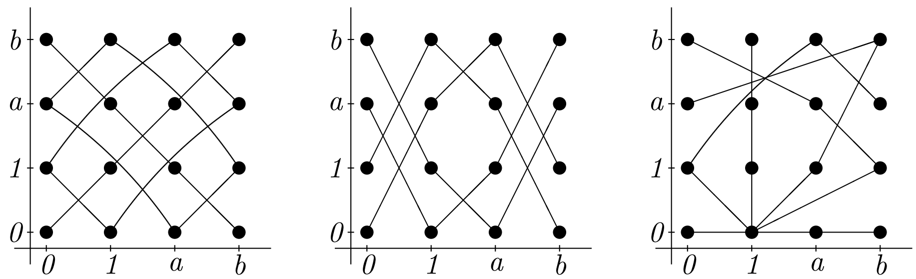
(i) *the horizontal lines: $y = 0$, $y = 1$, $y = a$ and $y = b$.*

(ii) *the vertical lines: $x = 0$, $x = 1$, $x = a$ and $x = b$.*

(iii) *the line of slope 1: $y = x$, $y = x + 1$, $y = x + a$ and $y = x + b$ (these are graphed on the left below).*

(iv) *the line of slope a: $y = ax$, $y = ax + 1$, $y = ax + a$ and $y = ax + b$ (these are graphed in the center below).*

(v) *the line of slope b: $y = bx$, $y = bx + 1$, $y = bx + a$ and $y = bx + b$ (not graphed).*



There are 48 dilatations.

(i) *16 dilations with "magnification" a:*
$\sigma_{h,k,a}(i,j) = (a(i-h)+h, a(j-k)+k)) = (ai+bh, aj+bk)$.
The traces of $\sigma_{0,1,a}$ are pictured on the right above.

(ii) *16 dilations with magnification b:*
$\sigma_{h,k,a}(i,j) = (bi+ah, bj+ak)$.

(iii) *16 translations $\tau_{h,k}(i,j) = (i+h, j+k)$.*
τ_{00} *is the identity with all lines for traces;*
τ_{01}, τ_{0a} *and* τ_{0b}, *have the vertical lines for traces;*
τ_{10}, τ_{a0} *and* τ_{b0}, *have the horizontal lines for traces;*
τ_{11}, τ_{aa} *and* τ_{bb}, *traces graphed on the left above;*
τ_{1a}, τ_{ab} *and* τ_{b1}, *traces graphed in the center above;*
τ_{1b}, τ_{ba} *and* τ_{a1}, *traces are not graphed.*

NOTATION. The subset of all non-degenerate dilatations is denoted by $\mathcal{D}$; the identity is considered a translation and the set of all translations is denoted by $\mathcal{T}$.

LEMMA 2.3.26a. *Let $\Theta = (P, L)$ be an affine plane. A translation $\tau \in \mathcal{T}$ is uniquely determined by the image of any one point.*

PROOF. Let p be any point. Assume that $\tau(p) = p$. Since the only translation with a fixed point is the identity, $\tau = \iota$. Assume then that τ has no fixed points. Let q be any point not on the trace $\ell_{p\tau(p)}$. Then $\tau(q)$ must be the point of intersection of the line through $\tau(p)$ parallel to ℓ_{pq} and the trace $\ell_{q\tau(q)}$ (the line through q parallel to the trace $\ell_{p\tau(p)}$). Now using q in place of p we see that the images of the points on ℓ are also uniquely determined. □

THEOREM 2.3.26. *Let $\Theta = (P, L)$ be an affine plane. The non-degenerate dilatations form a group $\mathcal{D}$ and the translations form an invariant (normal) subgroup $\mathcal{T}$. Furthermore,*

(i) *If σ is any dilatation and τ any translation other than ι, then τ and $\sigma\tau\sigma^{-1}$ have the same direction.*

(ii) *The identity and all translations with a given pencil of parallel lines as direction form a subgroup.*
(iii) *If translations with different directions exist, $\mathcal{T}$ is commutative.*

Proof. Since each dilatation is 1 to 1 and onto, $\mathcal{D}$ is a subset of the permutation group of P that contains the identity; hence we need only show that $\mathcal{D}$ is closed under composition and the taking of inverses. Let σ and ρ be two dilatations and p and q distinct points. Since σ is a dilatation $\ell_{\sigma(p)\sigma(q)}$ is parallel to ℓ_{pq} and, Since ρ is a dilatation, $\ell_{\rho\sigma(p)\rho\sigma(q)}$ is parallel to $\ell_{\sigma(p)\sigma(q)}$. By the transitivity of $\|$, $\ell_{\rho\sigma(p)\rho\sigma(q)}$ is parallel to ℓ_{pq} and $\rho\sigma$ is a dilatation.

Next consider σ^{-1}. Let $p' = \sigma^{-1}(p)$ and $q' = \sigma^{-1}(q)$. Since σ is a dilatation, $q = \sigma(q')$ is on the line through $p = \sigma(p')$ parallel to $\ell_{p'q'}$ that is $\ell_{pq}\|\ell_{p'q'}$. But then $\sigma^{-1}(q) = q'$ is on the line parallel to ℓ_{pq} through $\sigma^{-1}(p) = p'$ and σ^{-1} is a dilatation.

Let τ be a translation. If $\tau^{-1}(p) = p$, then $\tau(p) = \tau\tau^{-1}(p) = p$ and $\tau = \tau^{-1} = \iota$. We conclude that the inverse of a translation is the identity or has no fixed point, i.e. is a translation.

Now let τ_1 and τ_2 be translations and assume that $\tau_2\tau_1(p) = p$. Then $\tau_1(p) = \tau_2^{-1}\tau_2\tau_1(p) = \tau_2^{-1}(p)$ and we conclude that $\tau_2^{-1} = \tau_1$ and $\tau_2\tau_1 = \tau_2\tau_2^{-1} = \iota$. So $\tau_2\tau_1$ is the identity or has no fixed point, i.e. is a translation and $\mathcal{T}$ is a subgroup of $\mathcal{D}$.

Let σ be any dilatation, let τ be any translation and let $\tau' = \sigma\tau\sigma^{-1}$. Suppose that p is a fixed point for τ'. Then $\sigma\tau\sigma^{-1}(p) = p$ and applying σ^{-1} to both sides gives $\tau\sigma^{-1}(p) = \sigma^{-1}(p)$. Thus τ has $\sigma^{-1}(p)$ as a fixed point; which implies that $\tau = \iota$ and $\tau' = \iota$. We conclude that $\tau' = \iota$ or has no fixed point. In either case, τ' is a translation and so $\mathcal{T}$ is an invariant subgroup.

(i) Let p be any point, then $\ell = \ell_{\sigma^{-1}(p),\tau(\sigma^{-1}(p))}$ is a line in the direction of τ. Applying the dilatation σ to the point $\sigma^{-1}(p)$, give us that the line through $\sigma(\sigma^{-1}(p)) = p$ and $\sigma(\tau\sigma^{-1}(p))$ is parallel to ℓ. But the line $\ell_{p,\sigma\tau\sigma^{-1}(p)}$ is a trace line for τ'. Hence, τ and τ' have the same direction.
(ii) Let $\tau_1 \neq \iota$ and $\tau_2 \neq \iota$ be two translations with the same direction. Since τ_1 and τ_2 have the same direction, the trace $\ell_{p\tau_1(p)}$ is parallel to the trace $\ell_{p\tau_2(p)}$ and, since p is on both lines, they are the same line which we denote by ℓ. Since $\tau_1(p) \in \ell$ and ℓ is a trace of τ_2, $\tau_2\tau_1(p) \in \ell$ and, since p and $\tau_2\tau_1(p)$ are both on ℓ, ℓ is also a trace of $\tau_2\tau_1$. Hence $\tau_2\tau_1$ has the same direction as τ_1 and τ_2.

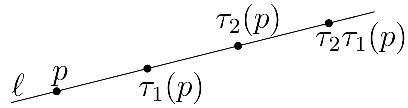


FIGURE 8

Finally, one easily checks that, for any translation $\tau \neq \iota$, $\tau^{-1}(p), p, \tau(p)$ all lie on the same line and so τ^{-1} and τ have the same direction.

(iii) Let $\tau_1 \neq \iota$ and $\tau_2 \neq \iota$ be two translations with different directions and let $p \in P$. Let ℓ_i denote the line $\ell_{p\tau_i(p)}$ and ℓ_i' denote the line through $\tau_j(p)$ parallel to ℓ_i, for $\{i, j\} = \{1, 2\}$. Note that ℓ_i' is a direction line for τ_i; hence ℓ_1' and ℓ_2' are not parallel and intersect in a point q.

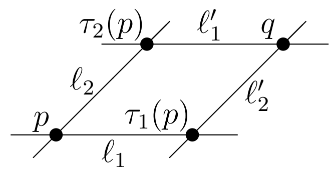


FIGURE 9

Since τ_2 is a dilatation, $\tau_2\tau_1(p)$ lies on ℓ_1'. Since ℓ_2' is a direction line for τ_2, $\tau_2\tau_1(p)$ lies on ℓ_2'. We conclude that $\tau_2\tau_1(p) = q$ and by symmetry that $\tau_1\tau_2(p) = q = \tau_2\tau_1(p)$.

Now assume that $\tau_1 \neq \iota$ and $\tau_2 \neq \iota$ are translations with same direction and let τ_3 be a third translation with a different direction. Suppose that $\tau_2\tau_3$ had the same direction as τ_1. But τ_2^{-1} has the same direction as τ_1 and so would $\tau_3 = \tau_2^{-1}(\tau_2\tau_3)$. Hence, τ_1 and $\tau_2\tau_3$ have different directions and therefore commute. We have:

$$(\tau_1\tau_2)\tau_3 = \tau_1(\tau_2\tau_3) = (\tau_2\tau_3)\tau_1 = \tau_2(\tau_3\tau_1) = \tau_2(\tau_1\tau_3) = (\tau_2\tau_1)\tau_3.$$

Composing both sides of $\tau_1\tau_2\tau_3 = \tau_2\tau_1\tau_3$ with τ_3^{-1} on the right yields $\tau_1\tau_2 = \tau_2\tau_1$.

□

THEOREM 2.3.27. *(Affine version of Desargues's Theorem) Let* $\Theta = (P, L)$ *be an affine plane constructed from a skew field. Let* $\ell_{qq'}$, $\ell_{rr'}$ *and* $\ell_{ss'}$ *be three lines that are either parallel or concurrent at a point distinct from these six points. If* $\ell_{qr} \| \ell_{q'r'}$ *and* $\ell_{qs} \| \ell_{q's'}$, *then* $\ell_{rs} \| \ell_{r's'}$.

DEFINITION 2.3.28. *An arbitrary affine plane* $\Theta = (P, L)$ *for which the conclusion of Desargues's Theorem holds is called a* *Desarguesian affine plane*

LEMMA 2.3.29a. *Let* $\Theta = (P, L)$ *be a Desarguesian affine plane.*

(i) *Given any two points* $p, p' \in P$, *there exists a translation* $\tau_{pp'}$ *that maps* p *onto* p'.
(ii) *Given collinear points* c, p, p', *there exists a dilation* σ *with fixed point* c *that maps* p *onto* p'.

PROOF.

(i) We start by defining the required translation for all points off of the line $\ell_{pp'}$. For any point r not on $\ell_{pp'}$, we define $\tau_{pp'}(r) = r'$ by the following construction: let ℓ_1 be the line through r parallel to $\ell_{pp'}$ and let ℓ_1' be the line through p' parallel to ℓ_{pr}. By the transitivity of $\|$, ℓ_1 and ℓ_1' cannot be parallel. Let r' be their point of intersection and define $\tau_{pp'}(r) = r'$ See Figure 10. Note that $r' \neq r$. Now repeat this construction for all points not on $\ell_{pp'}$. Hence, $\tau_{pp'}$ is well defined for all points not on $\ell_{pp'}$. Furthermore, it satisfies the definition of a dilatation for all points not on $\ell_{pp'}$ and fixes no point not on $\ell_{pp'}$.

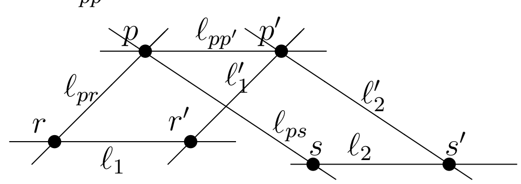


FIGURE 10

Using this same construction, we define $\tau_{rr'}$ for every point not on $\ell_{rr'}$ and note that $\tau_{rr'}(p) = p'$. Let s be any point not on $\ell_{pp'}$ nor on $\ell_{rr'}$ and let $s' = \tau_{pp'}(s)$. By the transitivity of parallelism, ℓ_2 the line through s parallel to $\ell_{pp'}$ is also parallel to $l_1 = \ell_{rr'}$; by Desargues's Theorem, we have that that $\ell_{rs} \| \ell_{r's'}$. Hence, $\tau_{rr'}(s) = s' = \tau_{pp'}(s)$. So, $\tau_{pp'}$ and $\tau_{rr'}$ agree at all points not on either of these parallel lines. Hence $\tau(q) = \begin{cases} \tau_{pp'}(q) & \text{when } q \text{ is not on } \ell_{pp'} \\ \tau_{rr'}(q) & \text{when } q \text{ is on } \ell_{pp'} \end{cases}$ is a well defined function taking p to p' that maps lines onto parallel lines and has no fixed point, that is, a translation.

(ii) The proof of this part is based on the second version of Desargues Theorem and is entirely parallel to the proof of the first part. See Figure 11.

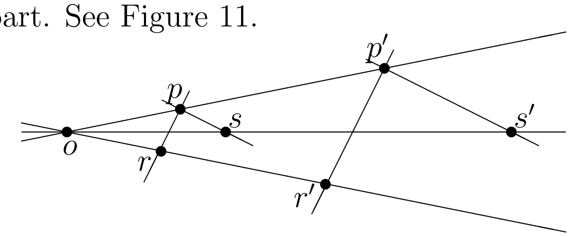


FIGURE 11

□

COROLLARY 2.3.29. *In a Desarguesian affine plane, the group of translations is commutative.*

DEFINITION 2.3.30. *A map $\alpha : \mathcal{T} \to \mathcal{T}$ is said to be* <u>*trace preserving*</u> *if*

(i) *α is a homomorphism, that is $\alpha(\tau_1\tau_2) = \alpha(\tau_1)\alpha(\tau_2)$ and*
(ii) *α preserves traces, that is either $\alpha(\tau) = \iota$ or τ and $\alpha(\tau)$ have the same direction.*

EXAMPLE 2.3.31. *For an examples of a trace preserving map, consider the maps $\alpha_0(\tau) = \iota = \tau^0$, $\alpha_1(\tau) = \tau = \tau^1$ and $\alpha_{-1}(\tau) = \tau^{-1}$. Indeed, for any integer m, $\alpha_m : \mathcal{T} \to \mathcal{T}$ defined by $\alpha_m(\tau) = \tau^m$ is a trace preserving map.*

Let $\Theta = (P, L)$ be the affine plane that comes from the field $\mathbb{K}$. The translations of Θ may be indexed by the two-dimensional vectors $\tau_{h,k}$, $h, k \in \mathbb{K}$ and are defined by $\tau_{h,k}(i, j) = (i + h, j + k)$. Then, exponentiation by any field element α, defined by ${\tau_{h,k}}^{\alpha} = \tau_{\alpha h, \alpha k}$ is always a trace preserving map. Consider τ^2 in Example 2.3.22 and τ^a in Example 2.3.25. We will eventually show that these are the only trace preserving maps.

NOTATION. In view of these examples, we will write τ^α instead of $\alpha(\tau)$ and identify the above trace preserving maps with the "field" elements -1 ,0, 1, . . . , m: $\tau \to \tau^{-1}$, $\tau \to \tau^0$, $\tau \to \tau^1$, . . . , $\tau \to \tau^m$. [Exponents will be read mod the "characteristic of the field."]

OBSERVATION. The trace preserving maps given by integer exponents are defined for every affine geometry. So we can easily recover the characteristic subfield for any Desarguesian affine geometry. In the case of Example 2.3.22, that is the entire field; in the case of Example 2.3.25, it is not. So there must be other trace preserving maps.

LEMMA 2.3.36a. *For any dilatation, σ, τ^α defined by $\tau^\alpha = \sigma\tau\sigma^{-1}$ is a trace preserving map.*

PROOF.

(i) α is a homomorphism:

$$(\tau_1\tau_2)^\alpha = \sigma\tau_1\tau_2\sigma^{-1} = \sigma\tau_1\sigma^{-1}\sigma\tau_2\sigma^{-1} = \tau_1^\alpha\tau_2^\alpha.$$

(ii) α preserves traces by Theorem 2.3.24 (i).

□

Note that if $\mathcal{T}$ is commutative and σ is a translation then α defined by $\tau^\alpha = \sigma\tau\sigma^{-1}$ is simply 1.

EXAMPLE 2.3.36. *In Example 2.3.25, we easily check that the trace preserving map given by $\sigma_{00a}\tau\sigma_{00a}^{-1}$ is τ^a:*
Recall $\sigma_{00a}(i,j) = (ai, aj)$ and $\sigma_{00a}^{-1} = \sigma_{00b}$. So

$$\sigma_{00a}\tau_{h,k}\sigma_{00b}(i,j) = \sigma_{00a}\tau_{h,k}(bi, bj) = \sigma_{00a}(bi+h, bj+k) =$$

$$(abi+ah, abj+ak) = (i+ah, j+ak) = \tau_{ah,ak}(i,j) = \tau^a(i,j)$$

DEFINITION 2.3.38.

(i) *The set of all trace preserving maps will be denoted by $\mathbb{K}$.*
(ii) *If $\alpha, \beta \in \mathbb{K}$, we may define $\alpha+\beta \in \mathbb{K}$ by $(\alpha+\beta)(\tau) = \alpha(\tau)\beta(\tau)$ for all $\tau \in \mathcal{T}$; in our notation $\tau^{\alpha+\beta} = \tau^\alpha\tau^\beta$.*
(iii) *Finally, the composition of trace preserving maps will be denoted by multiplication: $\tau^{\alpha\cdot\beta} = (\tau^\beta)^\alpha$.*

THEOREM 2.3.39. *Let $\Theta = (P, L)$ be a Desarguesian affine plane. If $\alpha, \beta \in \mathbb{K}$ then $\alpha+\beta, \alpha\cdot\beta \in \mathbb{K}$ and under these operations $\mathbb{K}$ is an associative ring with identity 1 (the identity trace preserving map).*

PROOF. We first show that $\mathbb{K}$ is closed under these two operations: Let α and β be two trace preserving maps. We consider the addition first.

(i) $\alpha+\beta$ is a homomorphism:

$$(\tau_1\tau_2)^{(\alpha+\beta)} = (\tau_1\tau_2)^\alpha(\tau_1\tau_2)^\beta = \tau_1^\alpha\tau_2^\alpha\tau_1^\beta\tau_2^\beta = \tau_1^\alpha\tau_1^\beta\tau_2^\alpha\tau_2^\beta = \tau_1^{(\alpha+\beta)}\tau_2^{(\alpha+\beta)}$$

(ii) $\alpha+\beta$ is trace preserving: If $\tau = \iota$ then $\tau^{(\alpha+\beta)} = \iota$. So assume $\tau \neq \iota$. Since α and β are trace preserving, τ^α and τ^β have the same direction as τ. By Theorem 2.3.26 (ii) $\tau^{(\alpha+\beta)} = \tau^\alpha\tau^\beta$ has this same traces as τ.
(iii) $\alpha\beta$ is a homomorphism:

$$(\tau_1\tau_2)^{(\alpha\beta)} = ((\tau_1\tau_2)^\beta)^\alpha = (\tau_1^\beta\tau_2^\beta)^\alpha = (\tau_1^\beta)^\alpha(\tau_2^\beta)^\alpha = \tau_1^{(\alpha\beta)}\tau_2^{(\alpha\beta)}$$

(iv) $\alpha\beta$ is trace preserving: Since β is trace preserving, the traces of τ are among the traces of τ^β; since α is trace preserving, the traces of τ^β are among the traces of $(\tau^\beta)^\alpha = \tau^{\alpha\beta}$.

Now we show that $\mathbb{K}$ is an associative ring with identity under these operations.

(i) Addition is associative:

$$\tau^{(\alpha+\beta)+\gamma} = \tau^{(\alpha+\beta)}\tau^{\gamma} = (\tau^{\alpha}\tau^{\beta})\tau^{\gamma} = \tau^{\alpha}(\tau^{\beta}\tau^{\gamma}) = \tau^{\alpha}\tau^{\beta+\gamma} = \tau^{\alpha+(\beta+\gamma)}$$

(ii) Addition is commutative:

$$\tau^{(\alpha+\beta)} = \tau^{\alpha}\tau^{\beta} = \tau^{\beta}\tau^{\alpha} = \tau^{(\beta+\alpha)}$$

(iii) The map $\tau^0 = \iota$ is the additive identity:

$$\tau^{0+\alpha} = \tau^0\tau^{\alpha} = \iota\tau^{\alpha} = \tau^{\alpha} = \tau^{\alpha}\iota = \tau^{\alpha}\tau^0 = \tau^{\alpha+0}$$

(iv) The map $\tau^{-\alpha} = (\tau^{\alpha})^{-1}$ is the additive inverse of τ^{α}:

$$\tau^{\alpha+(-\alpha)} = \tau^{\alpha}(\tau^{\alpha})^{-1} = \iota = \tau^0$$

(v) Left distributivity:

$$\tau^{(\beta+\gamma)\alpha} = (\tau^{\alpha})^{(\beta+\gamma)} = (\tau^{\alpha})^{\beta}(\tau^{\alpha})^{\gamma} = \tau^{(\beta\alpha)}\tau^{\gamma\alpha} = \tau^{(\beta\alpha+\gamma\alpha)}$$

(vi) Right distributivity:

$$\tau^{\alpha(\beta+\gamma)} = (\tau^{(\beta+\gamma)})^{\alpha} = (\tau^{\beta}\tau^{\gamma})^{\alpha} = (\tau^{\beta})^{\alpha}(\tau^{\gamma})^{\alpha} = \tau^{(\alpha\beta)}\tau^{(\alpha\gamma)} = \tau^{(\alpha\beta+\alpha\gamma)}$$

(vii) Multiplication is associative:

$$\tau^{(\alpha\beta)\gamma} = (\tau^{\gamma})^{(\alpha\beta)} = ((\tau^{\gamma})^{\beta})^{\alpha} = (\tau^{(\beta\gamma)})^{\alpha} = \tau^{\alpha(\beta\gamma)}$$

(viii) Multiplicative identity 1:

$$\tau^{1\alpha} = (\tau^{\alpha})^1 = \tau^{\alpha} \qquad \tau^{\alpha 1} = (\tau^1)^{\alpha} = \tau^{\alpha}$$

□

In order to prove the existence of multiplicative inverses, we need the following lemma.

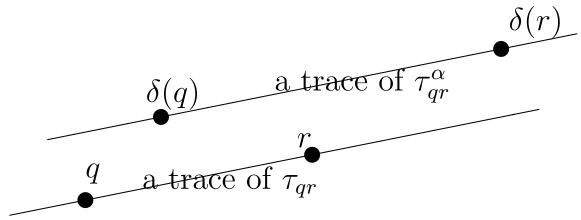
Lemma 2.3.41a. *Let $\Theta = (P, L)$ be a Desarguesian affine plane, let $\alpha \neq 0 \in \mathbb{K}$ and let p be any point in P. Then there exists a unique dilation δ that has p as fixed point such that $\tau^{\alpha} = \delta\tau\delta^{-1}$, for all $\tau \in \mathcal{T}$.*

Proof. Let $q \in P$ be any point and let τ_{pq} denote the translation that maps p onto q, Define $\delta(q) = \tau^{\alpha}_{pq}(p)$.

We have that $\tau_{qr}\tau_{pq} = \tau_{pr}$.

Applying α to both sides yields $\tau^{\alpha}_{qr}\tau^{\alpha}_{pq} = \tau^{\alpha}_{pr}$.

Evaluating both sides at p yields $\tau^{\alpha}_{qr}(\delta(q)) = \delta(p)$.



So ℓ_{pq} is a trace line of τ_{pq} while $\ell_{\delta(p)\delta(q)}$ is a trace line of τ_{pq}^{α} and, since α is a trace preserving map, $\ell_{pq} \| \ell_{\delta(p)\delta(q)}$. Hence δ is a dilatation. We also have that $\delta(p) = \tau_{pp}^{\alpha}(p) = \iota^{\alpha}(p) = p$. Suppose that δ were degenerate: $\delta(q) = p$ for some $q \neq p$. Then $\tau_{pq}^{\alpha}(p) = p$ and $\tau_{pq}^{\alpha} = \iota$ for all q. But since any translation $\tau = \tau_{p\tau(p)}$, $\tau^{\alpha} = \iota$ for all τ and $\alpha = 0$. So, under the assumption that $\alpha \neq 0$, δ defined by $\delta(q) = \tau_{pq}^{\alpha}(p)$ is a dilation with p as fixed point.

Finally, for any q,

we have $\quad \delta(q) = \tau_{pq}^{\alpha}(p) = \tau_{pq}^{\alpha}(\delta(p))$

which gives $\quad q = \delta^{-1}\tau_{pq}^{\alpha}\delta(p)$.

So, $\quad \delta^{-1}\tau_{pq}^{\alpha}\delta = \tau_{pq}$

or $\quad \tau_{pq}^{\alpha} = \delta\tau_{pq}\delta^{-1}$

From which we conclude that $\tau^{\alpha} = \delta\tau\delta^{-1}$, for any translation τ. □

THEOREM 2.3.41. $\mathbb{K}$ *is a skew field.*

PROOF. All that is left to show is the existance of multiplicative inverses for non-zero elements of $\mathbb{K}$. Let $\alpha \in \mathbb{K}$ with $\alpha \neq 0$. By the lemma, $\tau^{\alpha} = \delta\tau\delta^{-1}$ for some dilation δ. Let $\tau^{\beta} = \delta^{-1}\tau\delta$. Then

$$\tau^{\alpha\beta} = (\tau^{\beta})^{\alpha} = \delta(\delta^{-1}\tau\delta)\delta^{-1} = \tau = \tau^{1} \quad \textit{and}$$

$$\tau^{\beta\alpha} = (\tau^{\alpha})^{\beta} = \delta^{-1}(\delta\tau\delta^{-1})\delta = \tau = \tau^{1}$$

□

Now that we have constructed the skew field $\mathbb{K}$ from the Desarguesian affine plane $\Theta = (P, L)$, we must show that the affine plane constructed from $\mathbb{K}$ is indeed $\Theta = (P, L)$!

LEMMA 2.3.42a.

(i) *If* $\tau^{\alpha} = \iota$, *for some* $\tau \neq \iota$, *then* $\alpha = 0$; *if* $\tau^{\alpha} = \tau^{\beta}$ *for some* $\tau \neq \iota$, *then* $\alpha = \beta$.
(ii) *If* $\tau_1 \neq \iota$ *and* $\tau_2 \neq \iota$ *are translations with the same direction, then there exists and* $\alpha \in \mathbb{K}$ *such that* $\tau_2 = \tau_1^{\alpha}$.

PROOF.

(i) Suppose that $\tau^\alpha = \iota$ and $\alpha \neq 0$. Apply α^{-1} to both sides to get $\tau = (\tau^\alpha)^{-\alpha} = \iota^{-\alpha} = \iota$. Now suppose that $\tau^\alpha = \tau^\beta$ where $\tau \neq \iota$. Then $\tau^{\alpha-\beta} = \iota$ and $\alpha - \beta = 0$ or $\alpha = \beta$.

(ii) Now let $\tau_1 \neq \iota$ and $\tau_2 \neq \iota$ be translations with the same direction. Select $o \in P$, let $p = \tau_1(o)$ and let $p' = \tau_2(o)$. If $p = p'$ then, by Lemma 2.3.26a, $\tau_2 = \tau_1 = \tau_1^1$. Assume that $p \neq p'$ and let δ be the dilation guaranteed by Lemma 2.3.29a (ii) that fixes o and maps p onto p'. Let α be the trace preserving map $\tau^\alpha = \delta\tau\delta^{-1}$. Then

$$\tau_1^\alpha(o) = \delta\tau_1\delta^{-1}(o) = \delta\tau_1(o) = \delta(p) = p' = \tau_2(o).$$

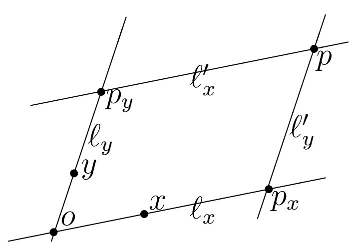
We conclude that $\tau_1^\alpha = \tau_2$.

□

Theorem 2.3.42. *Let $\Theta = (P, L)$ be a Desarguesian affine plane. Then the plane $\mathbb{K} \times \mathbb{K}$ is isomorphic to Θ.*

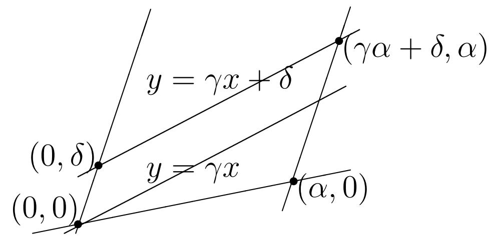
Proof. We start by coordinatizing Θ. Let o, x, and y denote three non-collinear points. We assign the coordinates $(0, 0)$, $(1, 0)$ and $(0, 1)$ to these three points, respectively. Let τ_x denote the translation that takes o to x and ℓ_x the the line through o and x; τ_y and ℓ_y are similarly defined.

Let p be any point, let $p_x = \ell_x \cap \ell'_y$ where ℓ'_y is the line through p parallel to ℓ_y and let $p_y = \ell'_x \cap \ell_y$ where ℓ'_x is the line through p parallel to ℓ_x. Next let $\alpha_x(p)$ be the element of $\mathbb{K}$ so that $\tau_x^{\alpha_x(p)}$ is the translation that maps o to p_x and let $\alpha_y(p)$ be the element of $\mathbb{K}$ so that $\tau_y^{\alpha_y(p)}$ is the translation that maps o to p_y. We then assign the coordinates $(\alpha_x(p), \alpha_y(p))$ to the point p. We easily check that the arbitrary element $(\alpha, \beta) \in \mathbb{K} \times \mathbb{K}$ is the pair of coordinates for the point $p = \tau_x^\alpha\tau_y^\beta(o) = \tau_y^\beta\tau_x^\alpha(o)$.



Assume that $\mathbb{K}$ is a field. Then all lines except the vertical lines may be given by an equation $y = \gamma x + \delta$. The solution set of this equation is the set of points of the form $\tau_y^{\gamma\alpha+\delta}\tau_x^\alpha(o)$ We may rewrite this expression:

$$\tau_y^{\gamma\alpha+\delta}\tau_x^\alpha(o) = \tau_y^{\gamma\alpha}\tau_x^\alpha\tau_y^\delta(o) = \tau_y^{\gamma\alpha}\tau_x^\alpha((0, \delta)) = (\tau_y^\gamma\tau_x)^\alpha((0, \delta))$$

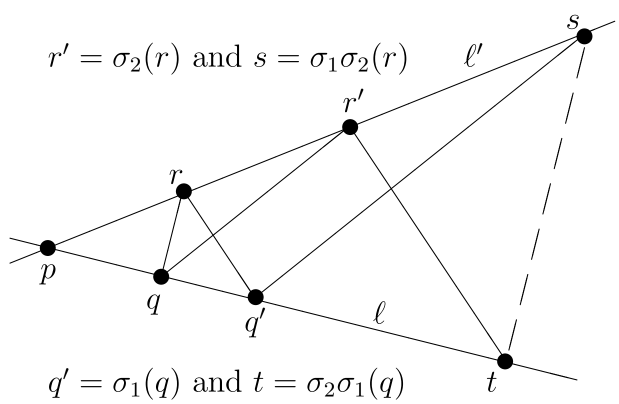


Since α is trace preserving, $(\tau_y^\gamma \tau_x)^\alpha((0,\delta))$ lies on the trace of $(\tau_y^\gamma \tau_x)$ that passes through $(0,\delta)$. So the points on the line of $\mathbb{K} \times \mathbb{K}$ given by $y = \gamma x + \delta$ all lie on the trace of $(\tau_y^\gamma \tau_x)$ passing through $(0,\delta)$ - a line of Θ. Now let p be any point on this trace line. Then the translation τ that maps $(0,\delta)$ onto p has the same direction as $\tau_y^\gamma \tau_x$ and by Lemma 2.3.42a (ii) must equal $(\tau_y^\gamma \tau_x)^\alpha$ for some $\alpha \in \mathbb{K}$. We conclude that this trace line of Θ and the line of $\mathbb{K} \times \mathbb{K}$ given by $y = \gamma x + \delta$ are the same. □

THEOREM 2.3.43 (Pappus - affine version). *Let $\Theta = (P, L)$ be an affine plane. If alternate vertices of a hexagon lie on intersecting lines such that two pairs of opposite sides are parallel, then the third pair of opposite sides are also parallel.*

THEOREM 2.3.44. *Let $\Theta = (P, L)$ be a Desarguesian affine plane. Then the skew field $\mathbb{K}$ is commutative if and only if Pappus' Theorm holds for Θ.*

PROOF. Let $p \in P$ be fixed and note that the dilations $\mathcal{D}_p$ with fixed point p form a group under composition. By Lemma 2.3.41a, this group is isomorphic to the multiplicative group $\mathbb{K} - 0$. Hence $\mathbb{K}$ will be a field if and only if $\mathcal{D}_p$ is a commutative group.



Let σ_1 and σ_2 be two dilations in $\mathcal{D}_p$. Let ℓ and ℓ'be distinct lines through p, let $q \neq p$ be a point on ℓ and let $r \neq p$ be a point on ℓ'. Next let $q' = \sigma_1(q)$, $r' = \sigma_2(r)$, $s = \sigma_1\sigma_2(r)$ and $t = \sigma_2\sigma_1(q)$. It follows that $\ell_{q's} \| \ell_{qr'}$ and $\ell_{r't} \| \ell_{rq'}$. We also have that $\ell t \sigma_2\sigma_1(r) \| \ell q, r$.

Hence $\sigma_2\sigma_1(r) = \sigma_1\sigma_2(r) = s$ if and only if $\ell t, s \| \ell q, r$. But $\sigma_2\sigma_1(p) = \sigma_1\sigma_2(p) = p$ so σ_1 and σ_2 will commute if and only if $\ell t, s \| \ell q, r$. $\square$