Why is that Box Empty?

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There are three basic counting formulas included in the high school curriculum. They can be organized as follows: we wish to count the number of ways to select *r* objects from a set of objects of *n* different types. In this general setting, there are two questions we need to ponder. *Are we ordering the objects? Can we select more than one object of a given type?* There are four ways to answering these questions, three of which lead to counting formulas we have all seen before.

Case 1: Suppose we desired to select *r* objects of *n* different types where the objects are ordered and repeats are allowed. The number of ways to do this is n^r.

A problem such as this is commonly structured as finding "words" of length *r*. There are 26 letters in the English alphabet. In this problem, we view this as 26 different types of letters and allow the letters to repeat. In our specific case $n = 26$ and suppose we want to make a word of length four, there are $26^4 = 456976$ possible "words."

Case 2: Suppose we desired to select *r* objects of *n* different types where the objects are ordered and repeats are not allowed. The number of ways to do this is $P(n, r) = \frac{n!}{(n-r)!}$.

These problems are often referred to as permuting elements. Consider the winners of a foot race, first place receives the gold medal, second place receives the silver medal, and third place receives the bronze medal. If there were 10 runners in the foot race, then there are ten choices for first place, and since no repeats are allowed, 9 for second place and 8 for third place. In this case, there are $10 \cdot 9 \cdot 8 = 720$ possibilities.

Case 3: Suppose we desired to select *r* objects of *n* different types where the objects are unordered and repeats are not allowed. The number of ways to do this is $C(n, r) = \frac{n!}{r!(n-r)!} = C(n, n-r)$.

We commonly refer to these problems as finding the total number of combinations. This can also be thought of as counting the number of subsets of size *r* in a set of *n* objects. For example the number of ways to select a committee of size 4 from a class of 25 students is $P(25,4) = \frac{25!}{4!(25-4)!} = \frac{25!}{4!\,21!} = \frac{25 \cdot 24 \cdot 23 \cdot 22}{4 \cdot 3 \cdot 2 \cdot 1} = 22650$

These counting problems are touched upon in the probability section of the school curriculum. The importance of these formulas lies not only in their connection to probability, but also in how they require students to think in an organized, systematic manner. The more we can train our students to organize external data and their own thoughts, the better prepared they are for the world. People are bombarded daily with data and predictions, it is necessary to equip our students with the ability to sort out this information and make informed choices and opinions. However, as we organized these counting problems, there is one more case that is completely ignored, even at the college level: **unordered with repeats allowed**. Why is this?

Could these counting problems be overlooked because there are no interesting examples? Well, consider the follow question you could use as a launch in your classroom, "Suppose you are in a candy store and want to buy a box of candy. How many different packages are possible containing 15 candies selected from 5 different flavors?" We could transfer this question into the algebraic question, "How many non-negative integer solutions $m_1, m_2, \ldots, m_n$ are there to the equation $m_1 + m_2 + \cdots + m_n = r$ where $n = 5$ and $r = 15$. Here m_1 is the number of candies of the first flavor, m_2 is the number of the second flavor, etc.

After gaining the students' attention with this question, allow them to investigate simple problems in order to prepare them to understand the general formula. For example, students can explore the number of non-negative integer solutions to the equation $m_1 + m_2 = 6$ (7 solutions). You can then move on to more difficult equations such as $m_1 + m_2 + m_3 = 6$ (28 solutions) or $m_1 + m_2 + m_3 + m_4 = 6$ (84 solutions).

As you can see there are interesting examples to provide in a classroom setting. Maybe this counting problem is not mentioned because the general solution is too difficult if approached unsystematically? However, the number of unordered r-selections from a set of n types with repeats is simply the combination $C(r+n-1,r) = C(r+n-1,n-1)$. For the example involving candy, it works out to be $C(15+5-1,5-1) = C(19,4) = 3876$. The students can easily check their solutions to the simpler problems against the formula given.

Perhaps these problems are skipped because it is too difficult to justify this formula? This formula is a bit more complicated than the first three, but not that much more complicated. Consider the following argument. When first introducing this counting problem you can bring in some chips or pennies and some dividers, such as straws or toothpicks. Explain to the students that if you are choosing r objects (the pennies or chips) from n different types, you can picture a solution by a string of pennies and straws:

p p … p s p p … p s … s p p … p

where there are m_1 pennies before the first straw (if $m_1 = 0$, start with a straw), m_2 pennies between the first and second straws (if $m_2 = 0$, the first and second straws are consecutive) and so on. There will be a total of r pennies and $n-1$ straws in this pictorial solution. It is easy to see that each bag of candy or solution to the algebraic problem can be pictured as a string of 15 (or r) pennies and 4 (or $n-1$) straws. It is also easy to see that each such string leads to a different bag of candy or a different solution to the algebraic problem. To find a particular solution, you need only select the $n-1$ positions for the straws out of the $r+n-1$ positions for the straws and pennies together. So the number of different bags candy or solution to the algebraic equation is $C(r+n-1,n-1) = C(19,4)$.

After teaching a unit with the more commonly known counting methods, you may want to use the following "Review Box" as way to launch into the topic of combinations with repeats allowed.

Select *r* objects of *n* types	No repeats	Repeats OK
Permutations	$P(n,r)$	n^r
Combinations	$C(n,r)$	

The box above organizes the selection of r objects from n different types under the following conditions: the selection is either ordered or unordered, repeats may be permitted or repeats may be forbidden. Of course, as soon as you put this table up, someone will ask "Why is that box empty?"

So, why IS that box usually left empty?