A Colorful Proof of Pick's Theorem

Jack E. Graver and Yvette A. Monachino

The *standard lattice* is the set of all points in the plane with integer coordinates. A polygon whose vertices are points of the lattice is called a *lattice polygon*. Pick's Theorem can be used to find the area of any lattice polygon.

Theorem. (*G. Pick, 1899*) Let $\mathbf{P}$ be a lattice polygon with I lattice points in its interior and B lattice points on its boundary. If A denotes the area of $\mathbf{P}$, then $A = I + B/2 - 1$. (See figure 1.)

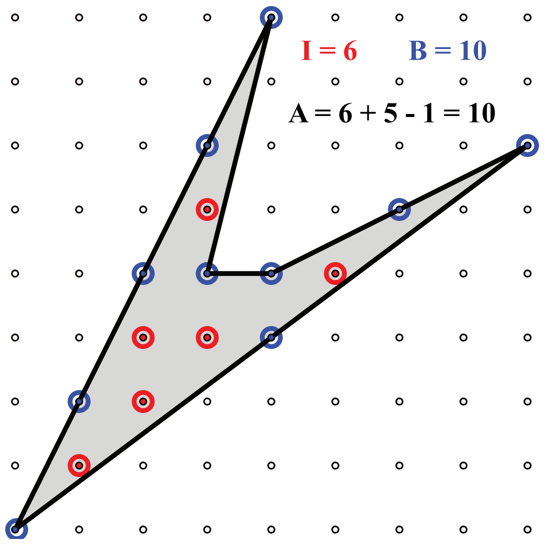

Figure 1.

There are many published proofs of this result, but almost all of them are constructed in the same basic way. They begin by showing that Pick's formula gives the area for some set of basic shapes; the set of all triangles, the set of right triangles and rectangles, and the set of triangles of area 1/2 are the three most commonly used sets. Once it is proven that the area of these basic shapes is given by Pick's formula, one shows that all lattice polygons can be constructed by adding and subtracting these shapes, and that the construction process respects Pick's formula. The paper by Grünbaum and Shephard (see the Further Reading section) includes a good set of references to these proofs. We offer a new proof in hopes that it will contribute to a deeper understanding of this important theorem.

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Before beginning the proof, we introduce a few concepts and definitions. The lattice that we are working with is not the usual lattice—it's the *dual* lattice that divides the plane into unit squares such that each unit square has a lattice point at its center. An *edge* is defined as a "side" of a unit square; when we use the term *side*, we are speaking about a side of the polygon. The sides of the polygon cut some of the squares into pieces, and we call those pieces that lie inside the polygon *fragments*. Fragments are categorized by the location of the lattice point. Fragments containing a lattice point in their interior are called *interior fragments*. If the lattice point in the fragment is located on the boundary of the polygon but is not at a vertex, then it is defined to be a *side fragment*. A fragment containing a vertex of the polygon is called a *vertex fragment*, and a fragment that contains no lattice point is an *ordinary fragment*. Examples of these fragments are labeled in figure 2.

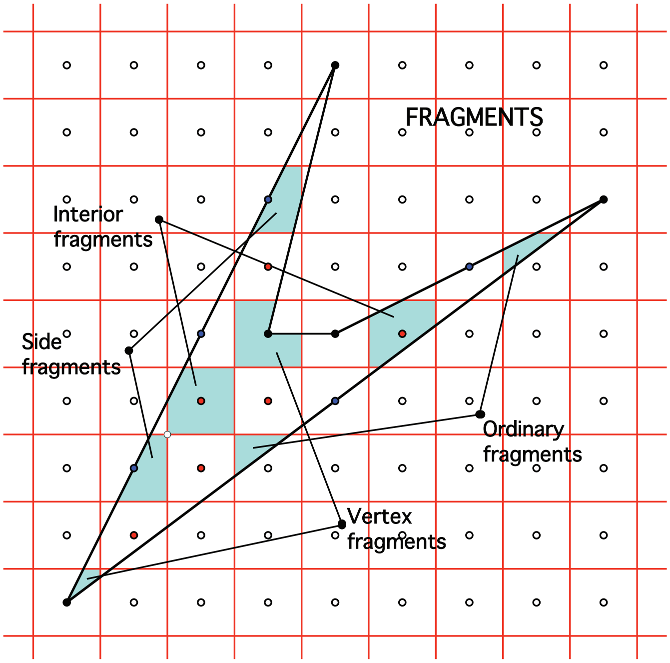

Figure 2.

A key observation is that *the area of the polygon is the sum of the areas of its fragments*.

Another concept crucial to our proof is that of matching fragments. Two fragments are *matched* if they share a boundary segment under a 180-degree rotation about the center of a side of the polygon. (See figure 3.) Because we are rotating around the center of a side, vertex fragments are matched to other vertex fragments. Furthermore, each vertex fragment is matched to its two neighboring vertex fragments when the vertices are ordered cyclically around the polygon. In his paper "A 'Natural' Approach to Pick's Theorem," Gordon Haigh introduced the dual lattice and used matching to prove Pick's formula for right triangles. But he then completed his proof of Pick's Theorem in the usual manner.

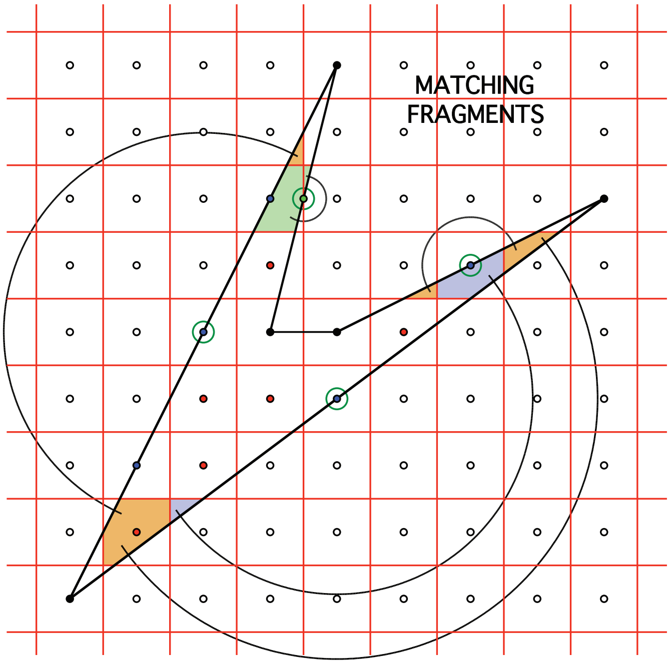

Figure 3.

Pick's Theorem makes a distinction between lattice points in the interior of the polygonal region and those on the boundary. Focusing on the boundary lattice points, we let V denote the number of *vertex boundary points* of the polygon and use $S = B - V$ for the remaining number of *side boundary points*. Thus, Pick's formula becomes $A = I + S/2 + V/2 - 1$, where, as above, I denotes the number of interior points.

The final concept we need for our proof is that of a coloring scheme for the fragments. The fragments containing lattice points are assigned distinct colors $c_1, c_2, \ldots, c_{I+S+V}$. Now we just have the ordinary fragments left to color. We do this by the following process: if an uncolored fragment is matched to a colored fragment, then it adopts the color of the latter. The process can be continued until all the fragments are colored. Of course, we must show that each fragment is assigned a color and that the assignments are independent of the order in which the fragments are colored.

With these concepts in hand, the proof is easy to outline. First, we color the vertex, side, and interior fragments with distinct colors. Next, the ordinary fragments are matched and colored, with the result being that each color class contains exactly one nonordinary fragment. Each color class containing an interior fragment can be assembled into a full square, contributing I square units to the total area. Each color class containing a side fragment can be assembled into a half square. Hence, the side fragments contribute $S/2$ to the total area.

Each vertex fragment is actually a *sector* of a square—a sector of a square being a region of the square between two rays from the center lattice point. The collection of vertex sectors can then be arranged to cover a square $V/2 - 1$ times. Taking all of the fragments together, we see that the total area is $I + S/2 + V/2 - 1 = I + B/2 - 1$. Figure 4 illustrates this process for our example. All that is left is to fill in a few details.

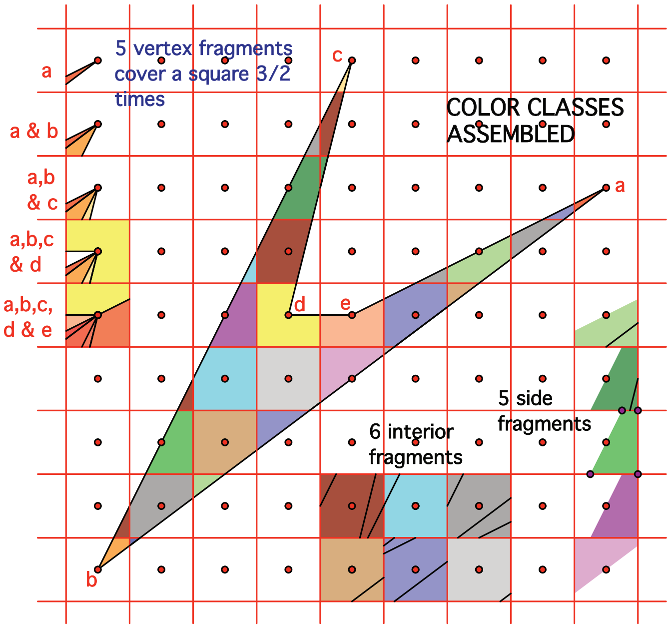

Figure 4.

Checking the Details

To prove Pick's Theorem, we have to show three things: (1) every fragment receives a unique color; (2) the fragments in a particular color class add up to a full square, a half square, or a sector of a square; and (3) the vertex sectors cover a square $(V/2 - 1)$ times. To do this, we need a better understanding of the half turns that match up fragments.

Let's first see why a half turn about the center of a side will always map the dual lattice onto itself. A side of our lattice polygon is a segment connecting two lattice points, so there are only a few possibilities for the coordinates of the center point of this segment. It could be another lattice point (if the "rise" and "run" of our segment are both even integers) or a vertex in the dual lattice (if the "rise" and "run" are both odd). Or it could fall on a midpoint of an edge (if the "rise" and "run" have different parity). In every case, it's straightforward to see that a 180-degree rotation around the center point not only takes the dual lattice onto itself, but also maps the boundary component along the side of a fragment onto a corresponding "matched" boundary component on the same side.

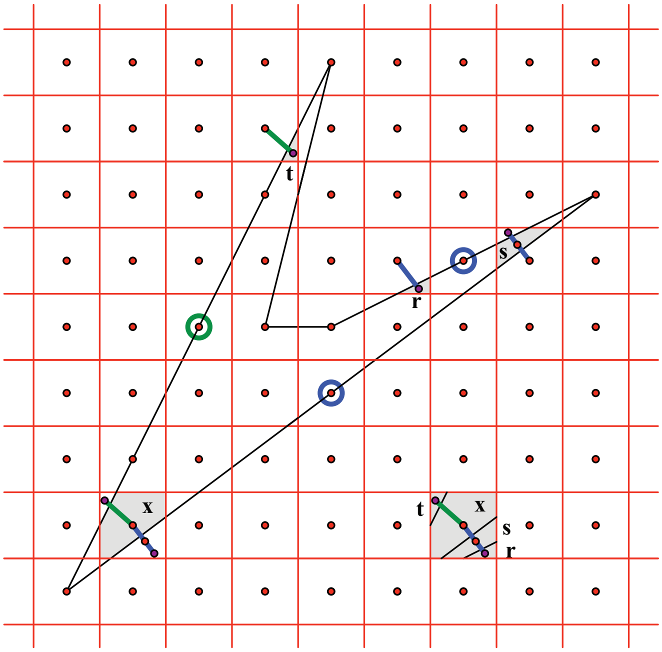

Figure 5.

Once the initial color assignments are made, only the ordinary fragments remain to be colored. To show that our algorithm gives a unique coloring of each ordinary fragment, we pick a point within an arbitrary ordinary fragment (such as point **r** or point **t** in figure 5) and draw the directed line segment from that point to the lattice point of the square that contains it. This segment must cross a side of the polygon and, in fact, may do so several times.

Starting from our selected point, consider the first side of **P** encountered along this segment, and note that this side is independent of our selected point. After a half turn about the midpoint of this side, the rotated image of the ordinary fragment "fits" into a square in the dual lattice. There is a natural image for the directed line segment, which still terminates at a lattice point. Continuing in the same direction, the rotated directed line segment must enter a new fragment, which is interior, side, or ordinary. If it is interior or side, we assign our original fragment the color of this fragment. If it is ordinary, we continue along our directed line segment until we (necessarily) encounter a new side of our polygon and repeat the matching process.

Because each successive half turn uses a different side from the one preceding it, the images of previous fragments will continue to fit with the new matched fragment, inside a single lattice square, and without any overlap. This implies that the process can't involve the same fragment twice (or else it would be periodic) and must eventually end after a finite number of steps at an interior or side fragment. When this occurs, all the fragments encountered en route inherit the color of the terminating fragment.

To show that the union of the fragments of a given color is a full square for an interior fragment, we reverse the above construction. Consider an interior fragment and select any point within the square containing it. This time, consider a directed segment from the center lattice point heading out to the selected point. Now it is possible to follow that segment—using successive half turns whenever a side of the polygon is encountered—until we arrive at a fragment containing (the image of) the selected point. The fragments encountered along the way must all be the same color, and as before, the rotated images of the fragments will fit into a single lattice square without overlap. Because the selected point was an arbitrary point in the original square, we conclude that the fragments of that color class can be assembled into a complete square.

If we start with a side fragment, a similar argument applies, provided our arbitrary point is chosen in such a way that the directed segment from the center lattice point passes through the interior of the side fragment. Following the same logic, we find that the fragments of this color class can be arranged into precisely half of a square.

Finally, recall that vertex fragments are always sectors of a square, and the sum of the interior angles of any polygon is $(V - 2) \times 180° = (V/2 - 1) \times 360°$. Hence, when added together, the vertex fragments cover a square $(V/2 - 1)$ times.

Further Reading

B. Grünbaum and G. C. Shephard, "Pick's Theorem," *American Mathematical Monthly* **100**(2) 1993, 150–161.

G. Haigh, "A 'Natural' Approach to Pick's Theorem," *Mathematical Gazette*, **64** 1980, 173–177.

About the authors: *Jack Graver is a professor of mathematics at Syracuse University working in both combinatorics and graph theory. Yvette Monachino is completing a master's degree in both mathematics and mathematics education at Syracuse University. The two have been giving workshops for math teachers in the Syracuse City Schools under Title II B Mathematics Science Partnership grant. This paper arose out of one of those workshops.*

email: jegraver@syr.edu
y.monachino@gmail.com

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