STUDENT RESEARCH PROJECTS

EDITOR

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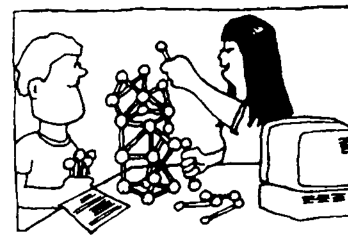

A student research project is an open-ended question or set of questions that is intended to give undergraduate students experience doing “junior” mathematical research. Readers are invited to share especially interesting and fruitful examples of such projects in this section. Describe the project in a form appropriate for presentation to the student investigator (normally no more than five double-spaced typed pages). Include appropriate references. On a separate page list the project title, the mathematical concepts involved in its investigation, and your affiliation. Send all proposals to Brigitte Servatius.

To further assist the editor in the evaluation of a project, provide a separate assessment of its difficulty and any information available about actual experience with the project. While untried projects are welcome, it should be clear that students can make genuine progress in their research.

Research Questions from Elementary Calculus

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Consider the following typical optimization problem from elementary calculus. A farmer wishes to construct two identical rectangular enclosures by dividing a single enclosure down the center. If the farmer has 120 feet of fencing, what are the dimensions of the enclosures of maximum area that can be constructed? One easily checks that the optimal enclosures are as in Figure 1.

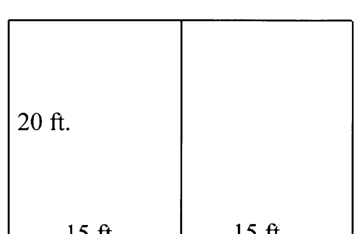


Figure 1.

Each enclosure of this optimal solution has area of $15 \times 20 = 300$ square feet. If the farmer wishes to construct two identical rectangular, 300 square foot enclosures side by side while minimizing the length of fencing used, the optimal configuration is the same. This represents a very general, but hardly new, observation about constrained optimization: reversing the roles of the constraint and objective functions frequently results in a problem with the "same" solution. Well known or not, it should be proved. And here, as in much research, finding the right level of generality and then properly formulating the result are integral parts, perhaps the most important parts, of the research. So, the first research problem in this research project is to formulate and prove this "dual optimization" principle. We continue our discussion assuming that an appropriate version of the dual optimization principle has been proved and simply refer to a configuration like the one pictured above as the optimal configuration without specifying which of the two optimization problems has been posed.

Referring to the above optimal configuration, we make a second observation: the amount of east-west fencing equals the amount of north-south fencing. This is true of the solutions to many such optimization problems. Consider the above problem but suppose that less expensive fencing can be used to divide the larger rectangle and that minimizing the cost to enclose a fixed area (or maximizing the area enclosed for a fixed cost) is the object. In this case, the optimal solution will be such that the *cost* of the east-west fencing equals the *cost* of the north-south fencing. For another example, consider the problem of constructing a rectangular enclosure along the side of a barn (see Figure 2). Assuming that the size of the enclosure is small relative to the side of the barn, the optimal solution will have half the fencing parallel to the side of the barn and half perpendicular to it. So we see that this "half and half" principle is valid in a variety of settings. Again, finding the right level of generality and properly formulating the result are essential parts of the project.

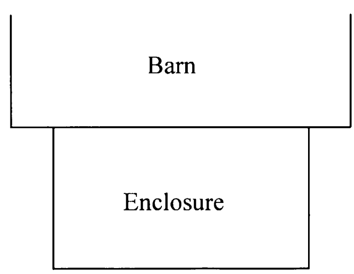


Figure 2.

Both of these principles have three-dimensional analogues which can be investigated. For example, consider building a chest using inexpensive wood for the bottom and back, moderately expensive wood for the sides and front, and expensive inlay for the top. Then maximizing volume for a fixed cost or minimizing cost for a fixed volume will yield the same optimal configuration (i.e., the same ratios between width, depth, and height) and that configuration will be the one in which the total cost for the top and bottom equals the total cost of the two sides which, in turn, equals the total cost of the front and back.

Later in this note we outline a proof of a limited formulation of these two principles. However they are valid in rather general settings and the first of the research projects we set forth is to state and prove these results at an appropriate level of generality:

Research Project 1. *State and prove a general formulation of the dual optimization principle and of the half and half principle.*

We note that, once these two results have been stated and proved, virtually all standard enclosure and box problems can be solved by a few simple algebraic steps.

Returning to two dimensions, we pursue another line of inquiry. Reconsider the problem of constructing two identical rectangular enclosures. If the basic structure is not constrained by the condition that it be constructed by dividing a single enclosure down the center, the solution remains the same but the problem is somewhat harder. One must first prove that an optimal solution occurs only when the two enclosures share a common side. The problem can be made even more interesting by requiring only that the enclosures be rectangular and have the same area. Again, by dropping the condition that the rectangles be congruent, the problem becomes a bit harder but the solution remains the same.

Increasing the number of rectangles leads to two very interesting collections of problems.

Problem $\mathbf{C_n}$. Find the optimal configuration for n congruent rectangular enclosures.

Problem $\mathbf{A_n}$. Find the optimal configuration for n rectangular enclosures of equal area.

Research Project 2. *Investigate the solutions to the problems* $\mathbf{C_n}$ *and* $\mathbf{A_n}$ *for all* n.

To give the reader a taste of this project, we sketch the solutions to $\mathbf{C_n}$ and $\mathbf{A_n}$ for $n = 2$ and $n = 3$. Investigating these problems will be greatly facilitated by the two principles described in Research Project 1 and we start this investigation by proving somewhat limited formulations of these principles. Suppose a configuration of n, $x \times y$ rectangles all with the same orientation has been selected. Problem $\mathbf{C_n}$ for this configuration can be formulated as:

$$\text{Maximize } a = xy \text{ subject to } rx + sy = f, \quad \text{or}$$

$$\text{Minimize } f = rx + sy \text{ subject to } xy = a,$$

where x and y are the dimensions of the congruent rectangular enclosures, r and s are constants which depend on the configuration, f is a fixed length of fencing and a the area or a is a fixed area and f is the length of fencing needed. In either case, the constraint equation permits us to think of y as a function of x. If we differentiate both equations with respect to x, we get

$$\frac{df}{dx} = r + s\frac{dy}{dx} \quad \text{and} \quad \frac{da}{dx} = y + x\frac{dy}{dx}.$$

Maximizing a for fixed f, we have $\frac{df}{dx} = 0$ and set $\frac{da}{dx} = 0$; minimizing f for fixed a, we have $\frac{da}{dx} = 0$ and set $\frac{df}{dx} = 0$. In either case, x and y must satisfy the system

$$r + s\frac{dy}{dx} = 0 \quad \text{and} \quad y + x\frac{dy}{dx} = 0.$$

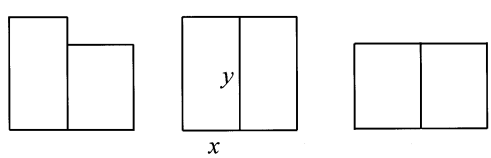
From this we conclude that, for each problem, the optimal configuration satisfies $rx = sy$. Thus, we have verified both principles for a special case of the problem $\mathbf{C_n}$, the case when all enclosures have the same orientation. Actually, the half and half principle need not hold for the general $\mathbf{C_n}$ problem but the dual optimization does hold for the general $\mathbf{C_n}$ problem. Assuming the dual optimization principle, we can restrict our investigation of problem $\mathbf{C_n}$ to a standard form: Minimize the total amount of fencing needed to construct n congruent rectangular enclosures each with an area of one square unit. For each possible configuration, we must

$$\text{Minimize } rx + sy \text{ subject to } xy = 1$$

and then select the configuration with the smallest minimum. By the half and half principle, the minimum will occur when $rx = sy$. Thus $rx = s\frac{1}{x}$, $x = \frac{\sqrt{s}}{\sqrt{r}}$, $y = \frac{\sqrt{r}}{\sqrt{s}}$ and $f = rx + sy$ has a minimum of $2\sqrt{rs}$. Armed with this information, the problem becomes a combinatorial/geometric problem of finding the configuration for which $2\sqrt{rs}$ has its smallest value.

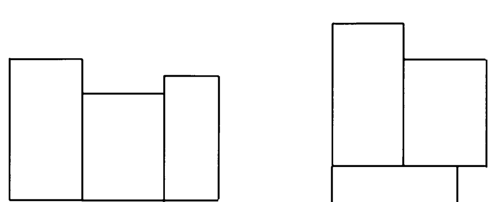
The discerning reader will have noted that we have made some basic assumptions about the optimal solutions to $\mathbf{C_n}$. First, we have assumed that all optimal configurations for $\mathbf{C_n}$ satisfy the condition that the sides of all rectangles are parallel to one of the axes of a fixed pair of orthogonal axes. Early in any investigation one would want to prove this for both $\mathbf{C_n}$ and $\mathbf{A_n}$. Second, it is easy to see that the constraint function has the form $rx + sy$ when at least one optimal configuration for $\mathbf{C_n}$ has all of its rectangles oriented in the same direction. One must show that the constraint function has this form even if some of the rectangles are "on their sides."

Moving on and assuming that our two principles hold for problem $\mathbf{A_n}$, we reformulate problem $\mathbf{A_2}$ in the standard format: minimize the fencing needed to enclose two rectangles of area 1 square unit each. Clearly, in the optimal configuration, the two rectangles will share one complete side of one of the rectangles:

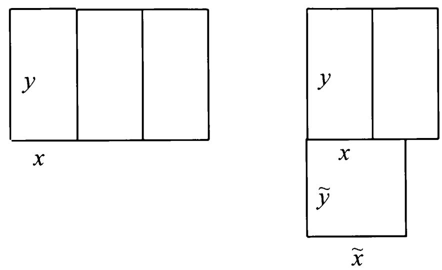


The left hand figure represents the general configuration and the right hand figures represent the two symmetric configurations constructed by reflecting each of the rectangles, in turn, about the common side. One easily checks that the average of the amount of fencing used in the two right hand figures is less than the amount of fencing used in the left hand figure. Thus, we may conclude that the optimal configuration will be symmetric about a common side and that the solution to problem $\mathbf{A_2}$ is the solution to problem $\mathbf{C_2}$. Labeling the central figure, we have that the amount of fencing is $f = 4x + 3y$; so $y = \frac{2}{\sqrt{3}}$, $x = \frac{\sqrt{3}}{2}$, and $f = 4\sqrt{3}$.

We note that, in general, if an optimal configuration for $\mathbf{A_n}$ consists of congruent rectangles, then it is also an optimal configuration for $\mathbf{C_n}$. Turning to problem $\mathbf{A_3}$, there are two basic configurations, three in a row or three in a cluster:



Using a sequence of symmetry arguments on pairs of enclosures, we can show that the three or two enclosures in a row must be congruent:



In the three in a row case, we have $f = 4y + 6x$ and conclude that the optimal dimensions for this configuration are $y = \frac{\sqrt{6}}{2} = \frac{\sqrt{3}}{\sqrt{2}}$, $x = \frac{2}{\sqrt{6}} = \frac{\sqrt{2}}{\sqrt{3}}$ and $f = 2\sqrt{24} = 4\sqrt{6}$. Turning to the cluster configuration, we first minimize the amount of fencing needed for the two identical enclosures. But that is simply the solution to $\mathbf{A_2}$: $y = \frac{2}{\sqrt{3}}$, $x = \frac{\sqrt{3}}{2}$ and $f = 4\sqrt{3}$. We now treat the third enclosure as if it were "built on the side of a barn." We have $\tilde{f} = \tilde{x} + 2\tilde{y}$; so $\tilde{y} = \frac{1}{\sqrt{2}}$, $\tilde{x} = \sqrt{2}$, and $\tilde{f} = 2\sqrt{2}$. So the total fencing needed for the right hand configuration is $f + \tilde{f} = 4\sqrt{3} + 2\sqrt{2}$, which is slightly less than $4\sqrt{6}$ (9.757 vs. 9.798).

The left hand configuration is a solution to problem $\mathbf{C_3}$ while the right hand configuration (the optimal solution to $\mathbf{A_3}$) is not. Before we can conclude that the left hand configuration is the optimal solution to problem $\mathbf{C_3}$, must consider the cluster configuration with the additional condition that all three rectangles are congruent. This leads to two cases: $\tilde{x} = x$ and $\tilde{y} = y$ or $\tilde{x} = y$ and $\tilde{y} = x$. In the first case, the total amount of fencing is $f = 5x + 5y$ and the optimal solution for this configuration has $x = y = 1$ and $f = 10$. In the second case, the total amount of fencing is $f = 6x + 4y$ and the optimal solution for this configuration is the same as the three in a row configuration. Thus, there are many optimal configurations for problem $\mathbf{C_3}$, all of which involve three $\frac{\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{3}}$ rectangles. In one case, they are in a row and, in the others, one end rectangle is removed and repositioned to form a cluster.

Moving on to $n = 4$, it is not surprising that problems $\mathbf{C_4}$ and $\mathbf{A_4}$ have a common solution, the 2×2 grid of unit squares. In fact, a natural conjecture is that, when $n = m^2$, problems $\mathbf{C_n}$ and $\mathbf{A_n}$ have a common solution in the $m \times m$ grid of unit squares. Proving this result might be a good place to start an attack on the general $\mathbf{C_n}$ and $\mathbf{A_n}$ problems.

There is one last natural extension of the $\mathbf{A_n}$ we wish to consider. Let the real numbers $r_1 \geq r_2 \geq \cdots \geq r_n > 0$ be given such that $r_1 + r_2 + \cdots + r_n = 1$. Maximize the total area a that can be divided into n rectangular regions of areas $r_i a$, for $i = 1, \ldots, n$, by a fixed amount of fencing. Or, given a fixed area a, minimize the amount of fencing needed to enclose n rectangular regions of areas $r_i a$, for $i = 1, \ldots, n$. Assuming that the dual optimization principle holds in this case, we denote this last formulation with $a = 1$ as the $\mathbf{R_n}$ problem. To illustrate this class of problems, we consider $\mathbf{R_2}$.

To facilitate our discussion of $\mathbf{R_2}$, we prove a variation on the "half and half" principle. Suppose we wish to minimize $f = a_1 x_1 + b_1 y_1 + a_2 x_2 + b_2 y_2$, where $x_i y_i = r_i$. Completing the square, we see that $a_i x_i + b_i y_i = (\sqrt{a_i x_i} - \sqrt{b_i y_i})^2 + \sqrt{a_i b_i r_i}$ and

$$f = \left(\sqrt{a_1 x_1} - \sqrt{b_1 y_1}\right)^2 + \left(\sqrt{a_2 x_2} - \sqrt{b_2 y_2}\right)^2 + \sqrt{a_1 b_1 r_1} + \sqrt{a_2 b_2 r_2}.$$

Thus, f is minimized when $a_1 x_1 = b_1 y_1$ and $a_2 x_2 = b_2 y_2$.

Now suppose that we have an optimal solution to $\mathbf{R_2}$. Clearly the two enclosures must share an entire side of one of them. Assuming that the dimensions of the enclosures are x_1 by y_1 and x_2 by y_2 and that the shared side is one in the y direction, we have that the amount of fencing to be minimized is $f = 2x_1 + 2x_2 + y_1 + y_2 + \max\{y_1, y_2\}$, where $x_1 y_1 = r_1$ and $x_2 y_2 = r_2$. If $y_1 < y_2$, the minimum occurs when $y_1 = 2x_1$ and $2y_2 = 2x_2$ or $y_1 = \sqrt{2r_1}$ and $y_2 = \sqrt{r_2}$. But, this minimum is in the given region only if $r_1 < \frac{r_2}{2}$. Under our assumption ($r_1 \geq r_2$), the minimum does not occur when $y_1 < y_2$. However, reversing the subscripts, we see that the minimum does occur at $y_1 = \sqrt{r_1}$ and $y_2 = \sqrt{2r_2}$ when $r_2 < \frac{r_1}{2}$. In the case that $\frac{1}{2} \geq r_2 \geq \frac{r_1}{2}$, the minimum must occur when $y_1 = y_2$. In this case, we have a one parameter problem, see Figure 3. By the "half and half" principle: $3y = 2(\frac{r_1}{y} + \frac{r_2}{y}) = \frac{2}{y}$; giving $y = \sqrt{\frac{2}{3}}$.

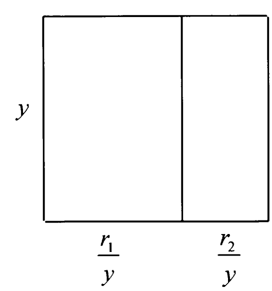


Figure 3.

Research Project 3. *Investigate the solutions to the problems* $\mathbf{R_n}$.

One might think of problem $\mathbf{R_n}$ as the rectilinear version of the two-dimensional soap film problem.

We close this note by observing that all of these research projects have interesting and challenging three-dimensional analogues.