Linear Functions and Rounding

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Introduction. Both linear functions and rounding functions ($\lfloor x \rfloor$, round down; $\lceil x \rceil$, round up; $\|x\| = \lfloor x+.5 \rfloor$, round to nearest integer) would likely be on anyone's list of the easiest functions to understand and use. Nevertheless when they are combined, surprising results and challenging problems arise, as we illustrate in this note.

Tax-Included Prices. Suppose that in the area where you live there is a 5% sales tax. Suppose further that you are selling souvenir jackets outside of a sporting event where it is difficult to make small change. So, you wish to advertise a price that includes tax and is an even dollar amount. Since your initial plan was to sell the jackets for \$18 each (\$18.90 with tax), you settle on \$19 as the ideal tax-included selling price. Let p denote the price before tax. You wish to solve $1.05p = 19.00$ for p. We have $\frac{19.00}{1.05} = 18.09523..$; so \$18.10 seems a reasonable choice for p. However, $(1.05)18.10 = 19.005$ which rounds to 19.01 while (1.05)18.09 rounds to \$18.99. Evidently \$19 is not a possible tax-included price!

A good place to start an investigation of rounded linear functions is with the problem: Given a t% sales tax find a formula for all amounts which can (or cannot) be tax-included prices. It will be helpful to reformulate the problem. Let $n \in \mathbf{N}$ ($\mathbf{N}$, the set of nonnegative integers) denote the price in cents. Assuming the tax t is given by a terminating decimal, $1 + .01t$ is a rational number, $\frac{p}{q}$. The tax-included prices are those integers in the image of the function $f : \mathbf{N} \to \mathbf{N}$, where $f(n) = \|\frac{p}{q}n\|$. The integers which are not tax-included prices are those in the complement of this image in $\mathbf{N}$. Our reformulated problem:

1. *For positive integers describe the image of f and its complement where $f : \mathbf{N} \to \mathbf{N}$ is given by $f(n) = \|\frac{p}{q}n\|$*

Solving this problem should give a good understanding of how rounded linear functions work and prepare for an investigation of the more complicated problems that follow.

Currency Exchange Rates. Another collection of rounded linear functions are the functions used to exchange one currency for another. Again, we will think of the functions involved in currency exchange as functions from $\mathbf{N}$ into $\mathbf{N}$ where the input variable is an integer amount of the smallest coin denomination of the first currency (cents, in the case of US currency) while the output is an integer amount of the smallest coin denomination of the second currency. But, here we will always round down to the nearest integer. For example on October 6, 1999 the exchange rate from US to Canadian currency was one US cent for 1.4718 Canadian cents. We can write this currency exchange function as $E(n) = \lfloor (1.4718)n \rfloor$. We define the general currency exchange function, $E_r(n)$, to be $\lfloor rn \rfloor$ where r is the rate of exchange. From an exchange function, one can construct a table of exchange values. The exchange rate just given yields the following table.

U.S. Cents	1	2	3	4	5	10	100	1000	10000
Can. Cents	1	2	4	5	7	14	147	1471	14718

Of course, a finite exchange table is only a partial view of an exchange function. The question we pose for investigation arises with the next table, which comes from a problem in a popular textbook where it is described as a table for converting US currency into German marks. The problem in the text is to find the exchange rate that gives the table.

Cents	500	1000	1500	2000	2500
Marks	1220	2440	3659	4879	6098

When this was assigned as a homework problem, most students concluded the rate was $r = 2.44$, which nearly works and agrees with the answer in the instructor's manual. However, one student looked into the problem more deeply, pointing out not only that $r = 2.44$ does not work but also that there is no rate that will give this table. To see this, note that from the second column $2440 \leq 1000r < 2441$ or $2.44 \leq r < 2.441$, while from the third column, $2.439\overline{3} \leq r < 2.44$. This might have been chalked up as a misprint in the problem and dropped, except that, the next day the same student observed that there is a rate, namely $s = 0.41$, for converting marks to cents which does give the entire table! This "one-way" table illustrates that our intuition can be in error. We might assume that, if s is a rate of exchange from marks to dollars, then $\frac{1}{s}$ is an exchange rate from dollars to marks and that "one-way" tables do not exist. Of course, $\frac{1}{s} = \frac{1}{0.41} = 2.43\overline{902439}$ almost works, and the problem is due to the rounding operation. Giving a precise explanation of this rather simple phenomenon requires some careful thought.

For our analysis, we will write currency exchange tables in the following form:

Currency M	m_1	m_2	m_3	m_4	$\cdots$	m_k
Currency N	n_1	n_2	n_3	n_4	$\cdots$	n_k

Exchange functions are non-decreasing, and we assume that in an exchange table both currency values are strictly increasing, i.e., $m_1 < m_2 < \cdots < m_k$ and $n_1 < n_2 < \cdots < n_k$. Any table of non-negative integers that satisfies these conditions will be called an exchange-type table. If it can be given by a rate of exchange function from currency M to currency N, we will call that rate of exchange an $(M \to N)$-rate; similarly, if it is given by a rate of exchange function from currency N to currency M, we will call that rate an $(N \to M)$-rate. An exchange-type table which is given by an $(M \to N)$-rate and an $(N \to M)$-rate is called a two-way table, one given by an $(M \to N)$-rate but not a $(N \to M)$-rate (or visa-versa) is called a one-way table and a table for which there is neither an $(M \to N)$-rate nor an $(N \to M)$-rate is called a no-way table.

It should be clear that any attack on the problem of one-way tables must be grounded on a good understanding of exchange functions. With this in mind we have arranged our research questions in three groups: First, some fundamental questions about exchange functions. Second, questions about exchange tables. And third, a single question, which the proposers have not completely solved, designed to lead the investigator into some of the deeper properties of exchange functions. While in practice all exchange rates are rational, the problems become more challenging if all positive real rates are considered.

Basic properties of exchange functions.

2. *For what rates r is $E_r(m)$ one-to-one?*

3. *For what rates r is $E_r(m)$ onto?*

4. *For what rates r and s is $E_r(m) = E_s(m)$ for all m?*

5. *For what rates r and s is $(E_r \circ E_s)(m) = m$ for all m?*

Properties of exchange tables.

6. *Let $\mathcal{T}$ denote an exchange type table and consider its subtables (obtained by keeping some columns and deleting others).*

(a) *Describe all $M \to N$ exchange rates for a 1-column subtable.*

(b) *Describe all $M \to N$ exchange rates for a 2-column subtable.*

(c) *Explain how a 2-column subtable may, or may not, admit an $M \to N$ exchange rate.*

(d) *Describe all $M \to N$ exchange rates for a k-column table.*

(e) *Is the U.S. cents to Canadian cents table one-way or two-way?*

7. *Suppose that we have an exchange-type table with k columns. Suppose further that each of the $\binom{k}{2}$, 2-column subtables is given by an $M \to N$ exchange rate. Must it follow that the entire table has an $M \to N$ exchange rate?*

8. (a) *Explain all 2-column one-way tables.*

   (b) *Explain all one-way tables.*

Further properties of exchange functions.

9. *For what rates r and s is there a rate t such that $(E_r \circ E_s)(m) = E_t(m)$, for all m?*

Batting Averages We introduce another surprising result involving linear functions and rounding with another problem from a textbook:

A baseball player comes to bat. If he gets a hit, his batting average will be .195 and, if he does not get a hit, it will be .190. Find how many times he has been at bat, how many hits he has had, and his present batting average.

To simplify our notation, we will express batting averages as three-digit whole numbers. If b stands for the number of times a player has been at bat and h for the number of hits the player has had, the player's batting average, expressed as a three-digit integer, is

$$\left\| 1000 \left(\frac{h}{b} \right) \right\|.$$

Here the round-off seems negligible compared with size of the numbers involved. So we will first attempt to solve the problem by ignoring the rounding function and solving the equations

$$1000 \left(\frac{h}{b+1} \right) = 190 \quad \text{and} \quad 1000 \left(\frac{h+1}{b+1} \right) = 195.$$

The solution $b = 199$ and $h = 38$, checks:

$$1000\frac{38}{199+1} = 190 \quad \text{and} \quad 1000\frac{38+1}{199+1} = 195,$$

with no rounding needed. The player's present batting average is:

$$\left\| 1000\left(\frac{38}{199}\right) \right\| = \|190.9547..\| = 191.$$

Now let's consider a more realistic problem – one in which the the numbers don't work out so nicely. Suppose that, if our player gets a hit, his batting average will be 325 and, if he does not get a hit, it will be 319. Our system of equations is

$$\left\| 1000\left(\frac{h}{b+1}\right) \right\| = 319 \quad \text{and} \quad \left\| 1000\left(\frac{h+1}{b+1}\right) \right\| = 325.$$

Solving this system without rounding yields $b = 165.\overline{6}$ and $h = 53.1\overline{6}$:

$$1000\frac{53\frac{1}{6}}{165\frac{2}{3}+1} = 319 \quad \text{and} \quad 1000\frac{53\frac{1}{6}+1}{165\frac{2}{3}+1} = 325.$$

But, b and h must be integers and the obvious choices are $b = 166$ and $h = 53$. However, these values do not check: $1000\frac{53}{166+1}$ rounds to 317 and $1000\frac{53+1}{166+1}$ rounds to 323, not the required 319 and 325. On the other hand, the natural second choice $b = 165$ and $h = 53$ does check: $1000\frac{53}{165+1} = 319.277108..$ and $1000\frac{53+1}{165+1} = 325.301204...$ Assuming this is the correct answer we compute his present average to be $1000\frac{53}{165} = 321.\overline{21}$ or 321. This all seems very reasonable until we look up the records to find that our player has actually had 61 hits in 190 times to bat! $1000\frac{61}{190+1} = 319.3717..$ and $1000\frac{61+1}{190+1} = 324.6073...$ This is rather unsettling particularly since this solution is very different from the exact solution to the system. Could there be other solutions? Can we even trust the answer to the textbook problem which worked out so nicely?

This leads to several more research questions:

10. *How many solutions can the batting averages problem have and how can we find them all?*

Returning to our second example, note that the present batting average computed from the incorrect solution was 321 and the player's actual present batting average is $\|1000\frac{51}{159}\| = 321$ also. Is this just luck or

11. *Will the present batting average be the same for all solutions?*

We close with a question designed to lead the investigator into some of the deeper properties of the functions involved.

12. *Letting b,h, T and t be continuous variables, investigate the transformation from the b,h-plane into the T,t-plane given by the system of equations:*

$$t = 1000\frac{h}{b+1} \quad \textit{and} \quad T = 1000\frac{h+1}{b+1}.$$

Also investigate its inverse. In particular, consider just how straight lines are mapped by these transformations. Discuss the solutions to the previous problems in the context of these transformations. Among other things, you should be able to estimate the expected number of solutions to a problem as a function of t and T.