The Linear Function on Review

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Preface

This book is intended as a bridge between high school level mathematics and college level mathematics. Its contents have been chosen to demonstrate the relationships between several topics from the high school mathematics curriculum and to look at those topics from a more mathematical sophisticated point of view. As the title indicates, the common thread of this work is the linear function. The three parts of the book are devoted, in turn, to: the linear function of a real variable, the linear function of a complex variable and the linear function of a 2-dimensional vector or a (2×1)-matrix variable. A second common thread is the visualization of these functions as transformations of one or two dimensional euclidean space.

It is the author's observation that the typical student's view of mathematics is that it is a collection of individual subjects each with its own collection of methods, techniques and results. The holistic approach of this book is intended to counter this view. The topics, examples and problems have all been chosen to encourage the integration of the major topics from the high school curriculum.

Another feature of this book is that it includes proofs. Not the formal "two column proofs" often included in the study of geometry in high school, but the more informal, but nevertheless rigorous, step by step explanation of why what we believe to be true is indeed true.

One of the common complaints of the colleges is that a large majority of incoming freshmen do not have algebra as a usable tool. In this text, algebra is seen to be an essential tool for the further study of mathematics. Algebraic skills are strengthened by constantly using algebra and by seeing the application of algebra in a variety of settings.

It is assumed that the reader will have a scientific calculator and will use it wherever appropriate.

As is apparent from the title, the concept of a function is central to this text and one of the major goals is to strengthen and broaden the readers understanding of this concept.

There are very few routine exercises in this book. The readers are encouraged to make up their own routine and nonroutine examples.

The idea that asking questions is as much a part of mathematics as answering them, is central to the philosophy of this text.

This book was written with four possible uses in mind. It could be used as a text in 12th year high school noncalculus course or as a text in freshman year college noncalculus course. In both cases, the purpose of such a course would be to strengthen the student's understanding of their high school mathematics and to upgrade their algebraic and geometric skills to tools that can be used in their college courses. A third use could be in the training of middle school and high school teachers. Again, the purpose is to strengthen the understanding of the algebra and geometry in the middle and high school curriculum. Finally, I hope that some high school students with a special interest in mathematics and some middle school and high school teachers will simply read this book because they find it interesting.

The first four chapters, devoted to a study of the real linear function, form the core of the book and could used as a stand alone unit in a high school or college level course. The first chapter introduces our way of interpreting real linear functions: as transformations of the number line. The geometric ramifications of this interpretation are subject of Chapter Two.

Our geometric understanding of linear functions enables us to give a very intuitive derivation of the formula for the n^{th} term of a solution to a linear difference equation. This is done in Chapter Three, where linear difference equations are discussed. In this chapter it becomes clear that the iteration of a linear function leads one naturally to the study of exponential function. One of the most useful and interesting applications of these exponential functions is to the mathematics of personal finance: compound interest, annuities and loans.

Chapter Four contains a variety of interesting applications of linear functions in business. Some quadratic functions arise here in a very natural way and enable us to review some of the algebra of quadratics. Also, some of fundamental applications of calculus arise. But, instead of using the derivatives, we use marginal functions: if $P(x)$ is the profit when producing x items, $P^*(x)$, the marginal profit, is defined to be the change in profit when one more item is produced. The marginal function is a very interesting and useful precalculus stand-in for the derivative.

Chapter Five, The Complex Linear Function, starts with an introduction to the complex numbers and ends with the equivalence between the complex linear functions and the direct congruences and similarities of the (complex) plane. Along the way, we review the algebra of complex numbers and some of the basics of trigonometric computations. We then we use this algebraic representation of geometric transformations to obtain some interesting geometric results.

Chapter Six starts with a geometric introduction to the two dimensional real vector space and the affine plane. After an introduction to matrix algebra, the geometric interpretation of the matrix linear function is introduced. This chapter ends with a classification of the similarities and congruences of the euclidean plane.

The book closes with a discussion of several applications of matrix algebra: systems of linear equations, to two-parameter linear difference equations and Markov processes. A big part of this chapter is a study of some theorems from euclidean geometry that are really affine results taking advantage of what we have discovered about the matrix linear functions that are not similarities and congruences of the euclidean plane.

There are very many people who must be thanked for the support they gave to me during the writing of this book. The first of which is my wife, Yana to whom the book is dedicated.

Some of the material here was designed for, and first used in, a New York State funded mathematics camp for high school students which was held at Syracuse University during the summer of 1992. Sixteen talented young people from all over New York State spent three weeks at Syracuse University exploring a wide variety of mathematical topics.

An expanded set of lectures on the linear function was then presented to a group of high school mathematics teachers in the program $MTRC^3$ (Mathematics Teacher/Researchers Collaborating for Collaboration in the Classroom). This program was funded by a National Science Foundation grant for four year period starting in the Spring of 1993. The project was designed to enable teachers to:

- develop collaborative classrooms that actively involve students in learning;
- promote mathematical thinking in secondary level mathematics classes;
- integrate technology into day-to-day instruction;
- sustain professional growth through teacher directed classroom research;
- insure ongoing professional development through participation in local professional organizations for mathematics teachers, professors and educators;
- function as teacher/mentors, building collaborative teams of mathematics teachers within each district.

Finally, several of individual topics evolved from lectures in a lecture program funded by the Syracuse University Mathematics Department. Under this program, faculty from SU visited four-year colleges in upstate New York giving colloquium lectures to their upper class mathematics students.

In addition to the support from NSF, the State of New York and Syracuse University there are many individuals who contributed to this project. Foremost among these are the three co-principal investigators of $MTRC^3$, Barbara Shelly, Patricia Tinto and Nancy Zarach. The program's administrative assistant, R. Deborah Davis, was most helpful as was Vincent Fatica. Several members of the faculty of the Syracuse University Department of Mathematics reviewed parts of the manuscript. And, of course, the reactions and many useful comments from the students and teachers who participated in afore mentioned programs were invaluable.

Members of the 1992 NYS Summer Mathematics Camp: Mathew Barker, Robert Chan, Jason Chiu, Eugene DeAngelis, Michael DeCresenzo, James Hamblin, Stephen Herrera, Amy Hetherington, Deana Renee Howell, Raul Kathirithamby, Joseph Kim, Maureen McCormick, Stacey Miller, Andre Newfield, Mara Weitzman, She-Chin Yeh.

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THIS IS A DRAFT!

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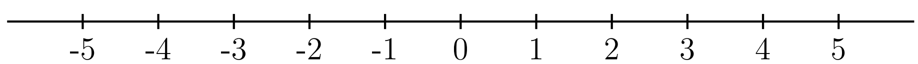
Contents

CHAPTER 1

The Real Linear Function

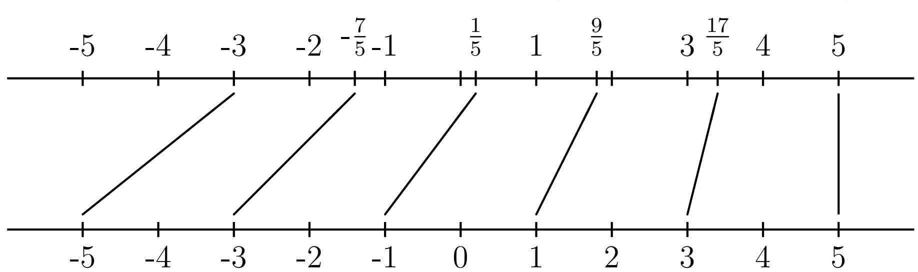
1.1. Picturing Linear Functions

Consider the real linear function $f(x) = \frac{4}{5}x + 1$. We will view this function and all real linear functions as a transformation of the real number line or simply the number line:

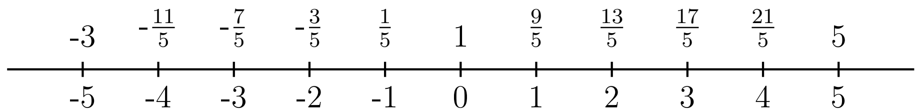


The feature of the number line that is critical to our discussion is that the scale is a *linear scale*, that is the distance on the line between two numbers is proportional to the absolute value of the difference of those two numbers. So, for instance, 2 and 4 are the same distance apart as $\frac{3}{2}$ and -$\frac{1}{2}$. In fact, we may use a linear scale to measure distance, defining the *measure of the distance* between two points x and y on the number line to be the absolute value of the difference of the two numbers: $|x - y|$.

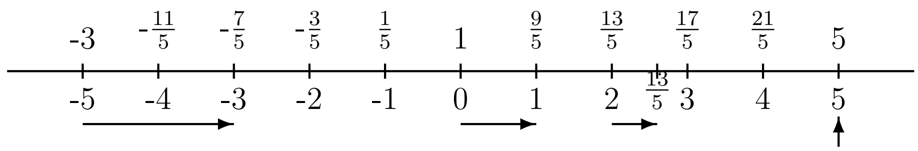
We start our investigation of the function $f(x) = \frac{4}{5}x + 1$ by drawing a picture. Instead of the usual geometric method of visualizing this function, the traditional graph of the function in a Cartesian coordinate system, we will view this function as map from one copy of the number line onto a second copy of the number line *(drawn above the first):*



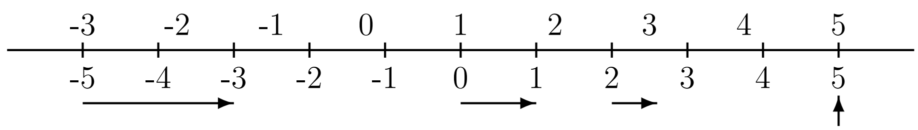
A yet simpler picture is obtained by writing the function values directly above the bottom line and omitting the top line:



We will call this way of picturing a function the *one-line (two-scale) graph* of the function. Using this graph, we will interpret f as a transformation of the line: the number 0 is mapped to 1, hence we think of 0 as being shifted to 1; similarly, -5 is shifted to -3, 2 to $\frac{13}{5}$, etc.

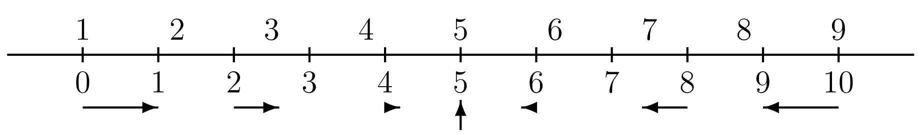


Observe that the set of function values of the integers seems to form a second linear scale for the line:

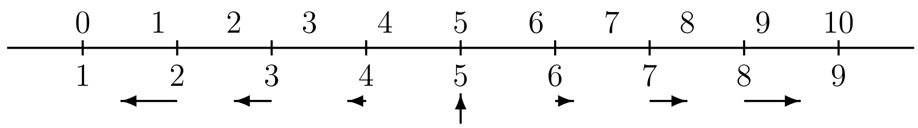


Note however, that these two linear scales measure distances differently. For example, the points labeled by 2 and 4 on the original scale are a distance 2 apart on that scale while they are labeled by $\frac{13}{5}$ and $\frac{21}{5}$ and are a distance $\frac{8}{5}$ apart on the second scale, a ratio of 1 to $\frac{4}{5}$. In fact, the ratio of the distances in the two scales is the same for all all pairs of points: given any two points on the line, the distance between them, as measured using the top scale, is $\frac{4}{5}$ of the distance between them, as measured using the original scale on the bottom. So, the effect of the function f is to shrink all distances by a factor of $\frac{4}{5}$.

Another feature of this one-line graph of f, is that the two scales seem to be getting closer together as one continues to the right. Shifting our "viewing window" to the right, we get:



From this viewpoint, we can see a very nice geometric description for f: all points are pulled in toward the "center" 5; specifically, the segment between each point and 5 is contracted to $\frac{4}{5}$ of its original size. We say that f is the "contraction" by a factor of $\frac{4}{5}$ about the point 5 (the center or "fixed point" of the contraction). Still another interesting property of the one-line graph, of a linear function is that the one-line graph of its inverse is obtained by simply interchanging the scales:



For the reader unfamiliar with the concept of the inverse of a function, we define it intuitively first: thinking of a function f geometrically as moving the points on the number line to other positions, we think of the inverse as the function which puts each point back in its original position - if such a function exists. The inverse of f is denoted by f^{-1} and algebraically out intuitive definition states that, if f^{-1} is the inverse of f, then $f^{-1}(f(x)) = x$, for all x. We will give a formal proof later that every linear function has a unique inverse.

EXERCISE 1.1. *Let* $f(x) = \frac{4}{5}x + 1$.

(i) *Verify that* $f^{-1}(x) = \frac{5}{4}x - \frac{5}{4}$, *by applying* $\frac{5}{4}x - \frac{5}{4}$ *to* $f(x)$.
(ii) *What is the inverse of* $\frac{5}{4}x - \frac{5}{4}$*?*
(iii) *Describe* $f^{-1}(x) = \frac{5}{4}x - \frac{5}{4}$ *geometrically.*

EXERCISE 1.2. .

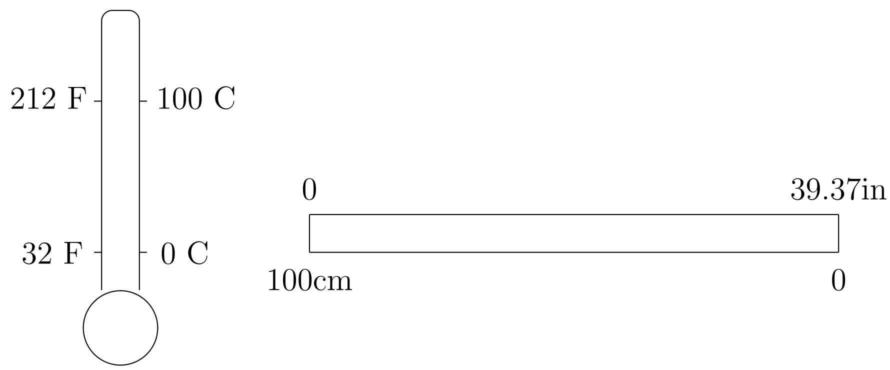
(i) *Draw the one-line graph of* $f(x) = \frac{3}{2}x + 2$ *and describe in a sentence its action on the line.*
(ii) *Compute the equation of the inverse of* $f(x) = \frac{3}{2}x + 2$ *and describe its action on the line.*

The next exercises arises from two common examples of one-line, two-scale graphs: a thermometer with both Fahrenheit and Centigrade scales and a meter stick with centimeters on one side and inches on the other. The two scale thermometer can be read in either direction, Centigrade to Fahrenheit or Fahrenheit to Centigrade, giving two linear functions which are inverses of one another. Similarly, the meter stick gives rise to a pair of functions.

EXERCISE 1.3. *Consider the thermometer described above and pictured below.*

(i) *Give the linear function* c *that converts the Fahrenheit measure of the temperature to its Centigrade measure.*
(ii) *Give the linear function* f *that converts the Centigrade measure of the temperature to its Fahrenheit measure.*
(iii) *Verify that these two linear functions are inverses of one another.*

(iv) *Find the temperature which has the same measure in the Fahrenheit and Centigrade scales.*
(v) *Describe both of these linear functions geometrically as transformations of the line.*



EXERCISE 1.4. *Consider the meterstick pictured above.*

(i) *Give the linear function f that has the meterstick as its one-line graph. (Note: this meter stich is designed so that each scale reads from left to right when it is on top. Hence, zero is not the fixed point.)*
(ii) *Find the fixed point of the meterstick, that is the point that has the same numerical measure in inches and centimeters.*
(iii) *Explain why f is not the function that converts centimeters to inches?*
(iv) *Find the function that does convert centimeters to inches.*

EXERCISE 1.5. *If the one-line graphs for a function is "turned on end", the result is simply a table of values for the function. Thus, a scientific calculator which produces tables of values for functions can be used to produce our one-line graphs.*

(i) *Enter the function for converting Fahrenheit and Centigrade into your calculator and make a conversion table.*
(ii) *Use the calculator to make a model of the meterstick.*
(iii) *Make a centimeters to inches conversion table.*

The examples discussed in this section are representative in that every linear function has a "nice" geometric description. It is this geometric interpretation of a linear function that we wish to explore.

To do so, we must have a precise definition of the geometry of the line and of the transformations of the line. That is our next task.

1.2. Basic Geometric Definitions

We will think of the real numbers as a coordinatized one-dimensional euclidean space - the *number line*. The *distance* between two real numbers x and y is given by $|x-y|$, the absolute value of $x-y$. In a euclidean space of any dimension, the distance is used to define the *euclidean transformations* of that space. Since the definitions of these euclidean transformations are the same in all dimensions, we give these definitions in the more general setting.

A mapping of a euclidean space (the line, euclidean plane, euclidean 3-space, euclidean 4-space, etc.) onto itself is called a *similarity* if all distances between pairs of points are altered (magnified or shrunk) by the same multiplicative factor, a positive real number, called the *magnification* of the similarity. Formally, the function f mapping a euclidean space to itself is a similarity with magnification m if the distance between $f(x)$ and $f(y)$ is m times the distance between x and y for all points x and y in the space. Specifically, for the line, the function f mapping the number line onto itself is a similarity with magnification m if $|f(x)-f(y)| = m|x-y|$ for all points x and y on the line.

We observed above that $f(x) = \frac{4}{5}x+1$ contracts all distances by $\frac{4}{5}$. We may verify by direct computation that f is, in deed, a similarity of the line with magnification $\frac{4}{5}$. Let x and y be any two real numbers, then:

$$|f(x)-f(y)| = |(\frac{4}{5}x+1)-(\frac{4}{5}y+1)| = |\frac{4}{5}(x-y)| = \frac{4}{5}|x-y|.$$

EXERCISE 1.6. *Verify by direct computation that the function f and its inverse from Exercise* **??** *are similarities.*

If the magnification of a similarity is 1, the similarity is called a *congruence*. As these terms indicate, a similarity will map a figure onto a similar figure and a congruence will map a figure onto a congruent figure. In fact, these transformations are the basis for the definitions of concepts "similar" and "congruent". Again the definitions are the same in all dimensions. Two sets S and T in a euclidean space are *similar* if there is a similarity f so that $T = f(S)$; two sets S and T in a euclidean space are *congruent* if there is a congruence f so that $T = f(S)$.

Our simple example leads to two natural questions: "Is every linear function a similarity?" and "Is every similarity of the number line given by a linear function?" We will answer the first of these two questions in a moment; the second of these two questions will be answered in the next chapter after an exploration of the geometry of the line.

EXERCISE 1.7.

(i) *Draw the one-line graph of* $f(x) = x - 3$ *and give a geometric description of this transformation of the line.*
(ii) *Draw the one-line graph of* $g(x) = -x+3$ *and give a geometric description of this transformation of the line.*
(iii) *Verify that the functions* f *and* g *are congruences of the number line.*
(iv) *Describe the inverses of* f *and* g *geometrically and then give their formulas.*

1.3. The Real Linear Functions

By a *linear function*, we mean a function of the form $f(x) = ax + b$ where a and b are fixed real numbers with $a \neq 0$ and x is a real variable. The constant functions $f(x) = b$ ($a = 0$) are sometimes included among the linear functions since their (traditional) graphs are straight lines. However, we will adopt the algebraic point of view and use the term "linear function" to denote the polynomial functions of degree one; hence, the constant functions are excluded. As is usual, a will be called the *slope* of $f(x) = ax + b$.

We start our formal discussion of the real linear functions by giving a formal proof that all linear functions are similarities of the number line.

LEMMA 1. *Let* $f(x) = ax + b$ *be a linear function. Then* f *is a similarity with magnification* $|a|$.

PROOF. We must show that f satisfies the definition of a similarity: Specifically, we must show that there is a positive real number m so that, for any two points x and y, the distance between $f(x)$ and $f(y)$ is m times the distance between x and y.

By direct computation:

$$\begin{aligned} |f(x) - f(y)| &= |(ax - b) - (ay - b)|, \\ &= |ax - ay|, \\ &= |a(x - y)|, \\ &= |a||x - y| \end{aligned}$$

Hence, f is a similarity with magnification $|a|$. □

It follows from this lemma that linear functions of the form $f(x) = x + b$ and $g(x) = -x + b$ are congruences of the number line. Geometrically, $f(x) = x + b$ moves every point on the line by a distance $|b|$ to the right, if $b > 0$, and to the left, when $b < 0$. When $b = 0$, $f(x) = x$ for all x and is called the *identity function.* We call a linear function of the form $f(x) = x + b$ a *translation*. It is convenient to adopt the notation $t_{[b]}(x) = x + b$ and the convention that $t_{[b]}$ is the translation by b units to the right, understanding that a translation to the right by a negative number to be a translation to the left. For example:

translation by 1 to the right, $t_{[1]}(x) = x + 1$;

-4	-3	-2	-1	0	1	2	3	4	5	6
-5	-4	-3	-2	-1	0	1	2	3	4	5

translation by $\frac{5}{4}$ to the left, $t_{[-\frac{5}{4}]}(x) = x - \frac{5}{4}$.

$-\frac{25}{4}$	$-\frac{21}{4}$	$-\frac{17}{4}$	$-\frac{13}{4}$	$-\frac{9}{4}$	$-\frac{5}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$	$\frac{7}{4}$	$\frac{11}{4}$	$\frac{15}{4}$
-5	-4	-3	-2	-1	0	1	2	3	4	5

In this notation, the identity function is the translation by 0: $t_{[0]}(x) = x$.

Geometrically, the linear functions of the form $g(x) = -x + b$ are a bit more complicated. Note that g interchanges 0 and b: $g(0) = b$ and $g(b) = 0$. Furthermore, it maps $\frac{b}{2}$ onto itself. Consider the example $f(x) = -x + 3$. This function maps $\frac{3}{2}$ onto itself as we see from its one-line graph:

8	7	6	5	4	3	2	$\frac{3}{2}$	1	0	-1	-2
-5	-4	-3	-2	-1	0	1	$\frac{3}{2}$	2	3	4	5

In this example, each point is flipped or reflected through the fixed point $\frac{3}{2}$: 1 is mapped to 2 and 2 is mapped to 1; 0 is mapped to 3 and 3 is mapped to 0. For any number w, we have $(\frac{3}{2} + w)$ is mapped onto $(\frac{3}{2} - w)$ while $(\frac{3}{2} - w)$ is mapped onto $(\frac{3}{2} + w)$. In general, $f(x) = -x + b$ has $\frac{b}{2}$ as fixed point and all points are reflected about this fixed point or center by the function. Specifically, $(\frac{b}{2} + w)$ is mapped onto $(\frac{b}{2} - w)$ while $(\frac{b}{2} - w)$ is mapped onto $(\frac{b}{2} + w)$, as you will now verify.

EXERCISE 1.8. *Consider the linear function $g(x) = -x + b$ and show that g interchanges points equidistant from $\frac{b}{2}$. Specifically, show* *that g leaves $\frac{b}{2}$ fixed and that, for all $w > 0$, g interchanges the points $\frac{b}{2} + w$ and $\frac{b}{2} - w$.*

We call a linear function of the form $g(x) = -x + b$ a *reflection*, specifically, *the reflection about* $\frac{b}{2}$. Replacing b by $2c$ in the formula, we adopt the notation $r_{[c]}(x) = -x + 2c$ and the terminology $r_{[c]}$ is the *reflection with center c.*

EXERCISE 1.9.

(i) *Give the linear function for the translation by $\frac{5}{2}$ units to the right; by 7 units to the left.*
(ii) *Give the linear function for the reflection about $\frac{5}{2}$.*
(iii) *Describe geometrically and give the linear function of the inverse for each of the congruences given on parts (i) and (ii) above.*

EXERCISE 1.10. *Describe the inverses of $t_{[b]}(x)$ and $r_{[c]}(x)$. Verify your answers.*

It is natural to ask if translations and reflections are the only congruences of the number line. By Lemma **??**, they are the only congruences which can be represented by a linear function. But, until we prove that that all congruences are given by a linear function, we cannot be sure that other congruences do not exist. We can only state that "we can't find any others" - hardly a convincing proof! Supplying a convincing proof that these are the only congruences is a fundamental feature of the next chapter. But, for now we continue our exploration of the properties of linear functions.

1.4. Compositions and Inverses

It will be convenient to have a general formula for the inverse function of a linear function. But first, we must give a formal definition of the inverse function and, to do this, we must define the composition of two functions. If $f(x)$ and $g(x)$ are any two functions, we may apply one and then the other to get a new function called the *composition* of f followed by g and denoted by $g \circ f$:

$$(g \circ f)(x) = g(f(x)) \quad \text{for each real number } x.$$

It is important to note that, in general, the composition of f followed by g $(g \circ f)$ and the composition of g followed by f $(f \circ g)$ are usually not the same function. For example let $f(x) = \frac{4}{5}x + 1$ and $g(x) = \frac{3}{2}x - 1$. Then:

$$(g \circ f)(x) = g(f(x)) = \frac{3}{2}(f(x)) - 1 = \frac{3}{2}(\frac{4}{5}x + 1) - 1 = \frac{6}{5}x + \frac{1}{2};$$

while

$$(f \circ g)(x) = f(g(x)) = \frac{4}{5}(g(x)) + 1 = \frac{4}{5}(\frac{3}{2}x - 1) + 1 = \frac{6}{5}x + \frac{1}{5}.$$

In the few instances where $g \circ f = f \circ g$, we say that f and g *commute*.

The fact that, in the above example, $(g \circ f)(x)$ and $(f \circ g)(x)$ are both linear functions is no accident. Let f and g denote arbitrary linear functions, say $f(x) = ax + b$ and $g(x) = a'x + b'$. Then

$$(g \circ f)(x) = a'(ax + b) + b' = (a'a)x + (a'b + b'),$$

which is a linear function. Since we will want to refer to this general fact later on, we will formalize it as a Proposition:

PROPOSITION 1. *Let f and g be linear functions. Then $g \circ f$ is a linear function. Specifically, if $f(x) = ax + b$ and $g(x) = a'x + b'$, then $(g \circ f)(x) = a''x + b''$ where $a'' = (a'a)$ and $b'' = a'b + b'$.*

EXERCISE 1.11. *Let $t_{[b]}$ and $t_{[b']}$ be two distinct translations ($b \neq b'$) and let $r_{[c]}$ and $r_{[c']}$ be two distinct reflections ($c \neq c'$).*

(i) *Show that $t_{[b]} \circ t_{[b']} = t_{[b']} \circ t_{[b]}$.*
(ii) *Show that $r_{[c]} \circ r_{[c']} \neq r_{[c']} \circ r_{[c]}$.*
(iii) *What interesting fact did you discover about the composition of translations?*
(iv) *What interesting fact did you discover about the composition of reflections?*
(v) *What's the next natural question to ask? What's the answer?*

Now we may formalize the concept of the inverse of a linear function. In our intuitive approach, we interpreted f geometrically and said that the inverse of f is the transformation of the line which returns each point to its original position. Restating this in terms of the composition of functions, we see that the function g is the *inverse* of f if $g \circ f = f \circ g$ is the identity function. Note that this definition is symmetric in the functions f and g. So, if g is the inverse of f, then f is the inverse of g. Actually, we should have defined g as *an* inverse of f - it is not at all obvious that f could not have two different inverses. In the next exercise you will show that indeed inverses are unique. Also, it is not at all obvious that the inverse of a linear function is another linear function. But, as we prove in the next proposition, the inverse of a linear function is a linear function and it is easy to compute

EXERCISE 1.12. *Prove that if g and h are both inverses of f, then $g = h$.*

PROPOSITION 2. *The linear function $f(x) = ax + b$ has, as it inverse, the linear function $g(x) = \frac{1}{a}x - \frac{b}{a}$.*

PROOF. That g is the inverse of f can be verified by direct computation:

$$f(g(x)) = a(\frac{1}{a}x - \frac{b}{a}) + b = x, \; for \; all \; x.$$

Similarly,

$$g(f(x)) = \frac{1}{a}(ax + b) - \frac{b}{a} = x, \; for \; all \; x.$$

□

As we noted earlier, the inverse of f is usually denoted by f^{-1}. Using this notation, $(f^{-1})^{-1} = f$!

EXERCISE 1.13. *Show that the inverse of a translation is a translation and that the inverse of a reflection is a reflection. Specifically, show both geometrically and algebraically that:*

$$t_{[b]}^{-1} = t_{[-b]} \quad \text{and} \quad r_{[c]}^{-1} = r_{[c]}.$$

EXERCISE 1.14. *Find all linear functions which are their own inverses.*

1.5. The Slope-Center Form

By a *fixed point* for a function $f(x)$, we mean a real number c so that $f(c) = c$. Translations other than the identity have no fixed point: all points are shifted. The surprising fact is that every linear function which is not a translation must have a fixed point! If the linear function is given by $f(x) = ax + b$, the question as to whether it has fixed point becomes: does the equation $x = ax + b$ have a solution? The answer is yes, if a is different from 1 or if $a = 1$ and $b = 0$. These observations are recorded in the next proposition and a formal proof of them is given.

PROPOSITION 3. *Let $f(x) = ax + b$.*

(i) *If $a = 1$ and $b = 0$, then f is the identity function and all points are fixed.*
(ii) *If $a = 1$ and $b \neq 0$, then f is a translation different from the identity and has no fixed points.*
(iii) *If $a \neq 1$, then f is not a translation and $\frac{b}{1-a}$ is its unique fixed point.*

PROOF. The real number x is a fixed point for f if and only if $x = ax + b$, that is, if and only if $(1 - a)x = b$.

(i) If $a = 1$ and $b = 0$, $(1 - a)x = 0 = b$, for all x, and every point is fixed. In this case, $f(x) = x$, which is the identity function.
(ii) If $a = 1$ and $b \neq 0$, $(1-a)x = 0 \neq b$, for all x, and so no point is fixed. In this case, $f(x) = x + b$, which is a translation different from the identity.
(iii) If $a \neq 1$, then f is not a translation. In this case, $(1-a)x = b$ if and only if $x = \frac{b}{1-a}$.

□

In view of the few examples we have already considered, we call the fixed point of a linear function its *center*. A nontranslation $f(x) = ax + b$ can be rewritten in *slope-center* form:

$$f(x) = ax + (1 - a)c,$$

where $(1 - a)c = b$ and $c = \frac{b}{1-a}$ is the center of f. In general, this form is important for an understanding of the geometric properties of the function; but, occasionally, a specific linear function has a natural presentation in the slope-center form. For example, a linear function popular among students is their final average as a function of their (yet to be taken) final exam. The center of this function is clearly their average at the end of their course work: if you have a 79% average on your course work and you get a 79% on your final exam, your final average will be 79%. Thus, it is easy to see that this linear function is given by $f(x) = ax + (1 - a)c$ where x is your score on the final exam (as a percent), c is your present average (as a percent) and a is the weight of the final exam (as a decimal).

EXERCISE 1.15. *Suppose that your present average is 79% and that the final exam accounts for two-fifth or 40% of your grade.*

(i) *Describe your final average as a function of the score x on your (yet to be taken) final exam.*
(ii) *What is the highest grade that you can get in this course? the lowest?*
(iii) *Compute the inverse of this linear function and explain just what it means.*

EXERCISE 1.16. *Let $f = ax + b$ be a linear function which is not a translation.*

(i) *Show that f and f^{-1} have the same center; explain this geometrically.*
(ii) *If $f(x) = ax+(1-a)c$ when written in slope-center form show that its inverse, in slope-center form, is $f^{-1} = \frac{1}{a}x + (1 - \frac{1}{a})c$.*

EXERCISE 1.17. *Let $f(x) = ax+(1-a)c$ and $g(x) = a'x+(1-a')c$ be two linear functions with the same center c.*
(i) *Put $g \circ f$ in slope-center form and observe that it also has c as its center.*
(ii) *Show that two linear functions with the same center commute ($g \circ f = f \circ g$).*

We noted above that, in general, one should not expect two function to commute. On the other hand, you have just shown that when they have the same center they do commute. We close this section and chapter by proving that this is the only case in which two linear functions commute.

PROPOSITION 4. *Two linear functions commute if and only if one (or more) of the following cases hold:*
(i) *one of the functions is the identity;*
(ii) *both of the functions are translations;*
(iii) *neither are translations and they have the same center.*

PROOF. In Exercise **??**, you proved that two translations commute; in Exercise **??**, you proved that two non-translations with the same center commute. And one easily checks that the identity, $t_{[0]}$, commutes with every linear function. Thus, if any of the three cases hold, the linear functions commute.

Now assume that $f(x) = ax + b$ and $g(x) = a'x + b'$ are two linear functions that commute. We must show that one of the three cases hold. We do this by first computing both compositions:

$$(f \circ g)(x) = a(a'x + b') + b = aa'x + ab' + b,$$
$$(g \circ f)(x) = a'(ax + b) + b' = a'ax + a'b + b';$$

and then setting them equal:

$$\begin{aligned} aa'x + ab' + b &= a'ax + a'b + b' \\ ab' + b &= a'b + b' \\ b - a'b &= b' - ab' \\ (1 - a')b &= (1 - a)b' \end{aligned}$$

Suppose that $(1-a') = 0$, i.e. g is a translation. Then either $(1-a) = 0$ or $b' = 0$. In the first instance, f is also a translation and case 2 holds; in the second instance, g is the identity and the first case holds. Similarly, if $(1-a)=0$, either case 1 or case 2 holds. We may assume then that both $(1-a')\neq 0$ and $(1-a)\neq 0$. Thus, neither f nor g are translations. Dividing through the last equality above by $(1-a')(1-a)$, we have:

$$\frac{b}{(1-a)} = \frac{b'}{(1-a')},$$

from which we conclude that f and g have the same center, that is, case 3 holds. □

1.6. Congruence and Similarity

Given a set X and a relation $x \sim y$ (x is related to y) among the elements of X, the relation is said to be an *equivalence relation* if the following three properties are satisfied:

(i) (reflexive) $x \sim x$, for all $x \in X$;
(ii) (symmetric) if $x \sim y$, then $y \sim x$, for all $x, y \in X$;
(iii) (transitive) if $x \sim y$ and $y \sim z$, then $x \sim z$, for all $x, y, z \in X$.

A collection of subsets $\mathcal{P} \subseteq \{Y | Y \subseteq X\}$ of a set X is a *partition of* X if

(i) the empty set does not belong to $\mathcal{P}$;
(ii) each element of X belongs to exactly one of the sets in $\mathcal{P}$.

In set theory notation this last condition is $\cup_{Y\in\mathcal{P}} Y = X$ and $Y \cap Z = \emptyset$ for all distinct Y and Z in $\mathcal{P}$. The subsets in $\mathcal{P}$ are called the *cells* of the partition. The the fundamental relation between equivalence relations and partitions is stated in the next proposition:

Proposition 5. *Let the set X be given. If $\sim$ is any equivalence relation on X, then there exists a unique partition $\mathcal{P}$ of X so that $x \sim y$ if and only if x and y belong to the same cell of the $\mathcal{P}$. Furthermore, if $\mathcal{P}$ is any partition of X, then there exists a unique equivalence relation $\sim$ on X so that $x \sim y$ if and only if x and y belong to the same cell of the $\mathcal{P}$.*

Proof. For $x \in X$ let $[x] = \{y : y \sim x\}$. We note first that when $x \sim y$ then $[x] = [y]$: $[x] \subseteq [y]$ by transitivity; $y \sim x$ by symmetry and then $[y] \subseteq [x]$ by transitivity. Next we note that if $[x] \cap [y] \neq \emptyset$, then $[x] = [y]$: if $z \in [x] \cap [y]$ then $x \sim z$ and $y \sim z$ and so $[x] = [z] = [y]$. Finally, $x \sim x$; so $x \in [x]$ and $[x] \neq \emptyset$. It follows that $\{[x] : x \in X\}$ (without including duplicates) is a collection of pairwise disjoint non-empty subsets whose union is X.

Conversely, let $\mathcal{P}$ be a partition of X and define $x \sim y$ if x and y belong to the same cell of the partition. One easily checks that $\sim$ is reflexive (x belongs to the same cell as x), symmetric (if y belongs to the same cell as x, then x belongs to the same cell as y) and transitive (if x belongs to the same cell as y and y belongs to the same cell as z, then x belongs to the same cell as z). □

PROPOSITION 6. *Congruence and similarity are both equivalence relations on the set of all geometric figures in the plane.*

PROOF. Two figures are congruent if there is a congruence the maps one onto the other. The identity map shows that each figure is congruent to itself (reflexive). Since each congruence has an inverse and the inverse of a congruence is a congruence, being congruent is reflexive. Now suppose that figure F is congruent to figure G and that figure G is congruent to figure H. Then there is a congruence f that maps F onto G and a congruence g that maps G onto H. But then the composition $g \circ f$ is a congruence and it maps F onto H (transitivity).

The proof that similarity is an equivalence relation is similar and left as an exercise for the reader. □

EXERCISE 1.18. *Complete this proof.*

Given an equivalence relation, the cells of the corresponding partition are called the *equivalence classes* of that equivalence relation. In view of Proposition **??**, all geometric figures in the plane are divided into *congruence classes* , the equivalence classes under congruence and also into *similarity classes* , the equivalence classes under similarity. For example all equilateral triangles form a similarity class while all equilateral triangles with side length 1 form a congruence class.

EXERCISE 1.19. *Describe how squares and rectangles are partitioned into congruence and similarity classes.*

CHAPTER 2

The Geometry of the Euclidean Line

2.1. The Euclidean Line

Since virtually all of our formal encounters with geometry have been with the euclidean plane or euclidean 3-space, we tend to think of the geometry of the number line as being trivial and uninteresting. In thinking this way we are wrong! While the number line is the simplest of the euclidean spaces, its structure is far from trivial and is, in fact, rather interesting.

The simplest geometric object on the number line, other than a single point, is the interval or segment. Intervals come in three "flavors": *closed intervals*, two points (numbers) and all points between them; *open intervals*, all points between (but not including) two points; and the *half-open intervals*, all points between two points including exactly one of the two end points. We will adopt the notation $[p, q]$, for the closed interval between p and q; (p, q), for the open interval; and $[p, q)$ and $(p, q]$, for the two half-open intervals with p and q as end points. In all cases, we assume that $p \leq q$. A square bracket "[" or "]" indicates that the endpoint is included in the interval and a parenthesis "(" or ")" indicates that that endpoint is not included. The lengths of the intervals $[p, q]$, $[p, q)$, $(p, q]$ and (p, q) are all equal to $|p - q| = q - p$.

Our intuition tells us that two intervals of the same length and type should be congruent. For example, the intervals $[7, 12)$ and $[-7, -2)$ both have length 5 and therefore should be congruent. To verify this, using our definition of congruence, we must find a congruence of the line which maps $[7, 12)$ and $[-7, -2)$. If we translate the line 14 units to the left, 7 will be translated onto -7, 12 will be translated onto -2 and the points between 7 and 12 will be translated onto the points between -7 and -2. Thus, the translation $t_{[-14]}(x) = x - 14$ (or its inverse) shows that the two intervals are congruent.

Next consider the intervals $[7, 12)$ and $(-7, -2]$. Again, they are both half-open intervals and have the same length so we assume that they are congruent. However, a translation won't work this time since the interval must be "turned around". The reflection $r_{[\frac{5}{2}]}(x) = -x + 5$, through the point $\frac{5}{2}$ will work: 7, which is $4\frac{1}{2}$ units to the right of $\frac{5}{2}$, is reflected onto -2, which is $4\frac{1}{2}$ units to the left of $\frac{5}{2}$; 12, which is $9\frac{1}{2}$ units to the right of $\frac{5}{2}$, is reflected onto -7, which is $9\frac{1}{2}$ units to the left of $\frac{5}{2}$. Also the points between 7 and 12 are reflected onto the points between -7 and -2.

EXERCISE 2.1. *Consider each of the 15 pairs of intervals from the following list of six intervals:*

$$[-1,3),\ (4,8),\ [6,10],\ (-10,-6),\ [8,12],\ (2,6].$$

In each case, decide if the two intervals are congruent. If they are congruent, find a congruence which maps one interval onto the other. Where possible, find a second congruence between the intervals.

EXERCISE 2.2. *Let p, q and r be real numbers and assume that r is positive.*

(i) *Describe the congruence which maps $[p, p+r)$ onto $[q, q+r)$.*
(ii) *Describe the congruence which maps $(p, p+r]$ onto $[q, q+r)$.*
(iii) *Describe both congruences which map $[p, p+r]$ onto $[q, q+r]$.*

EXERCISE 2.3. *Each interval is congruent to itself since it is mapped onto itself by the identity map $t_{[0]}$ which is a congruence. Which intervals can be mapped onto themselves by a congruence not equal to the identity? Describe the congruence.*

The set of all numbers greater than or equal to (less than or equal to) a fixed number, is called a *closed ray* and the set of all numbers greater than (less than) a fixed number, is called an *open ray*. The notation we adopt is:

$$[p,\infty) = \{x \mid x \geq p\}; \qquad (p,\infty) = \{x \mid x > p\};$$
$$(-\infty,p] = \{x \mid x \leq p\}; \qquad (-\infty,p) = \{x \mid x < p\}.$$

EXERCISE 2.4. *Show that two rays are congruent if they and only if they are either both open or both closed.*

EXERCISE 2.5. *Give a complete description of the congruence classes of segments and rays.*

Now consider the two intervals $[7,12)$ and $[21,36)$. The second interval is three times as long as the first, which may lead us to suspect that they are similar. Specifically, we would like to find a similarity of the line which maps $[7,12)$ onto $[21,36)$. Clearly, the magnification of this similarity will have to be 3. The most commonly known similarities are the "dilations" (formal definition to come in the next section). The easiest dilations to describe are those which have zero as the center of dilation: each point x is mapped onto mx where m is the magnification of the dilation. A dilation which shows these two intervals to be similar is the dilation with center 0 and magnification 3: 7 is mapped to 21, 12 to 36 and the points between 7 and 12 to the points between 21 and 36. In general, finding the appropriate similarity which demonstrates the that two segments are similar is not so easy. So we will defer the consideration of more complicated examples until we have developed a better understanding of similarities.

2.2. Similarities

In Chapter 1, we proved that all linear functions are similarities of the number line. At that time, we stated that the converse was true: all similarities of the number line are linear functions. This is not at all obvious; the fact that we can't think of any other similarities is not very convincing. In this section, we show that the converse is true. The proof involves several steps the first of which is to show that a similarity is uniquely determined by its effect on any pair of distinct points. We formalize this in the following key lemma.

LEMMA 2. *Let f and g be two similarities and let x and y be two distinct points. If $f(x) = g(x)$ and $f(y) = g(y)$, then $f = g$.*

PROOF. Let p denote the common image point of x, $p = f(x) = g(x)$, and q denote the common image point of y, $q = f(y) = g(y)$. We start the proof by demonstrating that f and g have the same magnification: Let m be the magnification of f. Then

$$|g(x) - g(y)| = |p - q| = |f(x) - f(y)| = m|x - y|.$$

Thus, m is also the magnification of g. Next, we make the simple observation that $p \neq q$. Since $x \neq y$ and $m \neq 0$, we have:

$$|p - q| = |f(x) - f(y)| = m|x - y| \neq 0,$$

and we conclude that $p \neq q$.

To show that $f = g$, we must show that $f(z) = g(z)$ for all real numbers z. Suppose that, for some z, $f(z) \neq g(z)$. We will show that this supposition leads to a contradiction and therefore cannot be true. Note that

$$|f(z) - p| = |f(z) - f(x)| = m|z - x| = |g(z) - g(x)| = |g(z) - p|,$$

that is, p is equally distant from $f(z)$ and $g(z)$. In short, we have shown that, if $f(z) \neq g(z)$, then p is the midpoint of the segment with $f(z)$ and $g(z)$ as endpoints. Replacing x by y in this argument, we conclude that q is also the midpoint of the segment joining $f(z)$ and $g(z)$. Thus, if $f(z) \neq g(z)$, then $p = q$; a contradiction! We conclude that, for all real numbers z, $f(z) = g(z)$ and that $f = g$. □

Combining this lemma with Proposition **??** from the previous chapter, we can state and prove the main result of this section:

THEOREM 1. *A function f is a similarity of the number line if and only if f is a linear function. The absolute value of the slope of the linear function is the magnification of the similarity. A function f is a congruence of the number line if and only if f is a linear function with slope ± 1.*

PROOF. By Proposition **??**, if f is a linear function, then it is a similarity. Now assume that f is a similarity with magnification m; we must show that f is given by a linear function. If we can construct a linear function g which agrees with f at two distinct points, then, by the Lemma **??**, we can conclude that $f = g$, that is, f is the linear function g. Let $b = f(0)$ and let $a = f(1) - f(0)$. Since f is a similarity with magnification m,

$$|a| = |f(1) - f(0)| = m|1 - 0| = m$$

and so, $a = \pm m \neq 0$. Since $a \neq 0$, we may define g to be the linear function $g(x) = ax + b$. Then g and f are two similarities such that $g(0) = b = f(0)$ and

$$g(1) = a(1) + b = a + b = (f(1) - f(0)) + f(0) = f(1).$$

By Lemma **??**, $f = g$, and we conclude that f is a linear function.

Since, the magnification m equals $|a|$, f will be a congruence if and only if $a = \pm 1$. □

It is now clear from this theorem that translations and reflections are the only congruences of the number line:

COROLLARY 1.1. *The only congruences of the number line are the translations and the reflections.*

Theorem **??** also gives us a powerful algebraic tool for the study of similarities. For example, consider the problem discussed near the end of the last section: find a similarity which demonstrates that the intervals $[7, 12)$ and $[21, 36)$ are similar. By the theorem, such a similarity must be given by a linear function. Suppose then that $f(x) = ax + b$ maps $[7, 12)$ onto $[21, 36)$. Clearly 7 must map onto 21 and 12 must map onto 36. Thus:

$$21 = f(7) = 7a + b \quad \text{and} \quad 36 = f(12) = 12a + b,$$

which is a system of two equations in the variables a and b. Solving gives $a = 3$ and $b = 0$; so, as we had guessed, $f(x) = 3x$ is the similarity. Of course, we could have chosen to demonstrate that the intervals are similar by mapping $[21, 36)$ onto $[7, 12)$. In this case, we would have gotten the inverse of f: $f^{-1}(x) = \frac{1}{3}x$.

Let's consider a more complicated example: find the similarities which map $[5, 6\frac{1}{2}]$ onto $[19, 23\frac{1}{2}]$. Since we may map 5 to 19 and $6\frac{1}{2}$ to $23\frac{1}{2}$ or 5 to $23\frac{1}{2}$ and $6\frac{1}{2}$ to 19, there are two possibilities. As above each possibility gives rise to a system of two equations in a and b:

$$\left\{5a + b = 19;\ \tfrac{13}{2}a + b = \tfrac{47}{2}\right\} \text{ and } \left\{5a + b = \tfrac{47}{2};\ \tfrac{13}{2}a + b = 19\right\}.$$

Solving the first system gives $f(x) = 3x + 4$ and solving the second gives $g(x) = -3x + 38\frac{1}{2}$.

EXERCISE 2.6. *For each of* $f(x) = 3x + 4$ *and* $g(x) = -3x + 38\frac{1}{2}$*:*

(i) *give its one-line graph and mark the intervals* $[5, 6\frac{1}{2}]$ *and* $[19, 23\frac{1}{2}]$*;*
(ii) *compute its fixed point or center;*
(iii) *describe it geometrically.*

EXERCISE 2.7. *Consider each pair of intervals from the following list of intervals:*

$$[-1, 5),\ (4, 18),\ [6, 10],\ (-10, -3),\ [8, 10],\ (2, 6\tfrac{1}{2}].$$

In each case, decide if the two intervals are similar. If they are similar, find a similarity between the two intervals. Where possible give a second similarity between the intervals.

EXERCISE 2.8. *Let* p, q, r *and* s *be real numbers with* r *and* s *positive.*

(i) *Describe the similarity which maps* $[p, p + r)$ *onto* $[q, q + s)$*.*
(ii) *Describe the similarity which maps* $[p, p + r)$ *onto* $(q, q + s]$*.*
(iii) *Describe both similarities which map* $(p, p+r)$ *onto* $(q, q+s)$*.*

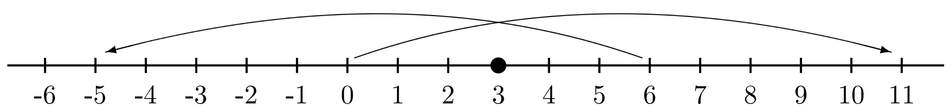
EXERCISE 2.9. *Give a complete description of the similarity classes of segments and rays.*

2.3. Classification of Similarities

It follows from Proposition **??** that all similarities except translations have a fixed point. Drawing the one-line graph with the fixed point in the center, motivates the following terminology. If $a > 0$ and $a \neq 1$, we call $d(x) = ax + b$ a *dilation*, specifically, the dilation with center $c = \frac{b}{1-a}$ and magnification a. Using the slope-center form, we adopt the notation $d_{[a,c]}(x) = ax + (1-a)c$ for the dilation with center c and magnification a. In this notation, the linear function $f(x) = \frac{4}{5}x + 1$, the dilation with center 5 ($5 = \frac{1}{1-\frac{4}{5}}$) and magnification $\frac{4}{5}$, is denoted by $d_{[\frac{4}{5},5]}(x)$. The term "dilation" indicates that the line is dilated from the center c by a factor equal to its magnification.

Later in our investigations, we will wish to distinguish between those dilations with magnification greater than one and those with magnification between zero and one. We call the former *expansions* and the latter *contractions* . Hence $d_{[\frac{1}{2},-4]}$ is the contraction about -4 by the factor $\frac{1}{2}$ and $d_{[\frac{5}{2},\frac{7}{4}]}$ is the expansion about $\frac{7}{4}$ by the factor $\frac{5}{2}$.

We extend the notation $d_{[a,c]}(x) = ax + (1-a)c$ for all linear functions with slope $a \neq \pm 1$ and center c. But how are we to interpret the similarity $d_{[a,c]}(x) = ax + (1-a)c$ when $a < 0$? Consider the linear function $d_{[-\frac{8}{3},3]}(x) = -\frac{8}{3}x + (1-(-\frac{8}{3}))3 = -\frac{8}{3}x + 11$. It not only stretches all lengths by a factor of $\frac{8}{3}$ but it also flips numbers across the fixed point 3. For example, a point at a distance 3 from the center is move to the point a distance 8 from the center on the on the other side of center:



Thus, it is the composition of the dilation $d_{[\frac{8}{3},3]}$ and the reflection $r_{[3]}$. But, in which order? Since they have the same center, the order makes no difference. We proved that at the very end of Chapter 1 in Proposition **??**. Hence

$$d_{[-\frac{8}{3},3]} = d_{[\frac{8}{3},3]} \circ r_{[3]} = r_{[3]} \circ d_{[\frac{8}{3},3]}.$$

EXERCISE 2.10. *Show that, for any a and c, $d_{[a,c]} = r_{[c]} \circ d_{[-a,c]}$.*

When $a < 0$ and $a \neq -1$, we call $f(x) = ax + (1-a)c = d_{[|a|,c]} \circ r_{[c]}$ the *dilating-reflection* with center c and magnification $|a| = -a$. We extend the notation $d_{[a,c]}(x) = ax + (1-a)c$ to include the case that a is negative: when $a < 0$, $d_{[a,c]}(x) = d_{[|a|,c]} \circ r_{[c]}$ and when $a = -1$, $d_{[a,c]}$ is simply the reflection $r_{[c]}$. It is now clear from Theorem **??** that dilations and dilating-reflections are the only similarities of the number line that are not congruences:

COROLLARY 1.2. *The only similarities of the number line which are not congruences are the dilations and the dilating-reflections.*

It follows from our classification of similarities that they naturally split into those that preserve direction and those that reverse direction. Translations and dilations map half-open intervals which open to the right onto half-open intervals which also open to the right while reflections and dilating-reflections map half-open intervals which open to the right onto half-open intervals which open to the left. We say that translations and dilations are *direct* similarities and that reflections and dilating-reflections are *opposite* similarities.

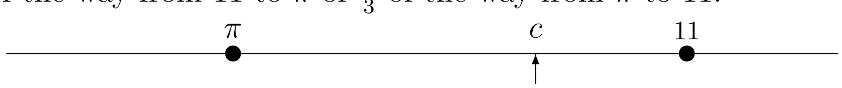
In the first section of this chapter, we considered the problem of identifying the similarity which mapped one interval onto another. Let us reconsider that problem in the light of what we have discovered about similarities. Suppose that we wish to identify the similarity f which maps $[5, 11)$ onto $(\pi, \pi + 12]$. By Theorem **??**, f is a linear function, that is $f(x) = ax + b$ and we need to compute a and b. What are our options? In Lemma **??**, we saw that a similarity is completely determined by its action on any two distinct points. So we could concentrate on the fact that $f(5) = \pi + 12$ and $f(11) = \pi$ and solve the system of two equations in two unknowns (a and b):

$$5a + b = \pi + 12 \quad \text{and} \quad 11a + b = \pi.$$

On the other hand, we could take advantage of what we can observe geometrically:

- $(\pi, \pi+12]$ is twice as long as $[5, 11)$, so the magnification is 2.
- f must reverse orientation; so, it is opposite and $a = -2$.
- Now we need only solve one equation, say $-22 + b = \pi$, to get $f(x) = -2x + 22 + \pi$.

Actually, we can push the geometrical reasoning even further. Since we have deduced that $a = -2$, f is a dilating-reflection and has the form $d_{[-2,c]}$. All that we have to do is identify c. On the number line, $d_{[-2,c]}$ reflects the segment [c,11] about and doubles its length, so c is $\frac{1}{3}$ of the way from 11 to π or $\frac{2}{3}$ of the way from π to 11.



We compute $c = \pi + \frac{2}{3}(11 - \pi) = \frac{22+\pi}{3}$.

EXERCISE 2.11. *Use geometric arguments to*

(i) *compute the similarity that maps*
   (a) $[7, 9)$ *onto* $[-7, 0)$,
   (b) 8 *onto* 12 *and* 4 *onto* 16,
   (c) $[5, \infty)$ *onto* $(-\infty, -5]$ *with a magnification of 2;*

(ii) *to compute the similarities that map*
   (a) $[7, 9]$ *onto* $[-7, 0]$,
   (b) 8 *onto* 12 *and has magnification* 3,
   (c) $[5, \infty)$ *onto* $(-\infty, -5]$ *with a magnification of 2.*

The wonderful thing about algebra is that we don't have to keep on solving such problems; we can just derive a formula!

LEMMA 3. *Given real numbers p, q and a positive real number m.*

(i) *If $m \neq 1$ then the two similarities that map p onto q are the dilation $d_{[m, \frac{q-mp}{1-m}]}$ and the dilating-reflection $d_{[-m, \frac{q+mp}{1+m}]}$.*
(ii) *If $m = 1$ then the two congruences that map p onto q are the translation $t_{[q-p]}$ and the reflection $r_{[\frac{q+p}{2}]}$.*

EXERCISE 2.12. *Prove this lemma.*

EXERCISE 2.13. *Let p and q be two distinct real numbers and let m be a positive real number.*

(i) *Find the locus of points x so that $|x - q| = m|x - p|$.*
(ii) *Relate the points of the locus to the similarities which map p onto q.*

EXERCISE 2.14. *Consider the traditional graph, in the plane, of a linear function $f(x)$. In terms of this graph:*

(i) *explain when f is a translation;*
(ii) *describe the center of f;*
(iii) *explain when f is a reflection;*
(iv) *explain when f is an expansion;*
(v) *explain when f is a contraction.*

2.4. The Algebra of Transformations

Given two similarities of the line it is natural to consider their composition. For example: what is the result of reflecting about 0 and then reflecting about 3? In general, how do we identify the composition of two congruences or similarities? Combining Theorem **??** and Lemma **??**, we have that the composition of two similarities is always another similarity and, combining the theorem with Lemma **??**, we have that the inverse of a similarity is another similarity. There are several other observations that we can make. For example, by Proposition **??**, the composition of two congruences is another congruence and, by Proposition **??**, the inverse of a congruence is another congruence. The next exercise contains a few more.

EXERCISE 2.15. *Explain why:*

(i) *the composition of two direct similarities is a direct similarity;*
(ii) *the composition of two opposite similarities is a direct similarity;*

(iii) *the composition of one direct and one opposite similarity, in either order, is an opposite similarity.*

By now the reader should have mastered all of the tools needed to investigate similarities of the number line on there own. It would be nice to leave the reader to discover his or her own results and then prove them. However, we will need to refer to many of these results later; so, we will give a formal statement of them here and leave the reader to prove them. We first consider the "algebra" of congruences. Note that we have already stated and proved several parts of this theorem; we included them here again for the sake of completeness.

THEOREM 2. *Consider the translations $t_{[b]}(x)$ and $t_{[b']}(x)$ and the reflections $r_{[c]}(x)$ and $r_{[c']}(x)$. Then*

(i) *The inverse of $t_{[b]}$ is the translation $t_{[-b]}$.*
(ii) *The reflection $r_{[c]}$ is its own inverse.*
(iii) *The composition of $t_{[b]}$ and $t_{[b']}$, in either order, is the translation $t_{[b+b']}$.*
(iv) *The composition $r_{[c]} \circ r_{[c']}$ is the translation $t_{[2(c-c')]}$.*
(v) *The composition $t_{[b]} \circ r_{[c]}$ is the reflection $r_{[c+\frac{b}{2}]}$.*
(vi) *The composition $r_{[c]} \circ t_{[b]}$ is the reflection $r_{[c-\frac{b}{2}]}$.*

EXERCISE 2.16. *Identify the references for those parts we have proved and prove the remaining parts.*

The "algebra" of arbitrary similarities is described in the next theorem.

THEOREM 3. *Let a, a', c and c' be real numbers with a and a' different from 0, 1 and -1. Then*

(i) $d_{[a,c]} \circ d_{[a',c]} = d_{[aa',c]}$;
(ii) $d_{[a,c]} \circ d_{[\frac{1}{a},c]} = d_{[1,c]}$, *the identity, hence* $d^{-1}_{[a,c]} = d_{[\frac{1}{a},c]}$;
(iii) $d_{[a,c]} \circ d_{[a',c']} = d_{[aa',c'']}$, *for some point c'' whenever $aa' \neq 1$;*
(iv) $d_{[a,c]} \circ d_{[\frac{1}{a},c']} = t_{[b]}$ *for some point b.*

EXERCISE 2.17. *Prove this theorem.*

EXERCISE 2.18. *The proof of Theorem* **??** *you just gave is undoubtedly algebraic. Now take time to think through each part geometrically.*

EXERCISE 2.19. *Compute the c'' in part (iii) and the b in part (iv).*

The set of all congruences of the line is "closed" under composition and the taking of inverses: the composition of two congruences is a congruence and the inverse of a congruence is a congruence. Any set of functions satisfying these two conditions is called a *group* . We will give a formal definition of group later; but, this specialized form of the definition will be sufficient for our purposes here. So, the set of all congruences is called the group of congruences. There are many subsets of the group of congruences which are also groups and to verify that a nonempty subset is a group, we need only verify that the inverse of any congruence in the set is in the set and that the composition of any two congruences in the set is also in the set. It follows from the theorem that the set of all translations is such a group, often called a *subgroup* since it is a group within another group. Note that the set of reflections is not a subgroup: it is closed under inverses but not under composition.

EXERCISE 2.20. *Which of the following collections of similarities are subgroups?*

(i) *The collection of all direct similarities.*
(ii) *The collection of all opposite similarities.*
(iii) *The collection of all similarities with the same center c.*
(iv) *The collection of all similarities with integral magnifications.*
(v) *The collection of all similarities with integral centers.*
(vi) *The collection of all translations by an integral amount.*

One of the sources of interesting subgroups are the subsets of the line: given a subset S, one may consider the collection of all congruences (or all similarities) which map S onto S. Clearly, the composition of two congruences (or similarities) in this collection is also in this collection and the inverse of any congruence (or similarity) in this collection is also in this collection. The collection of all similarities which map S onto S is called the *symmetry group of S*. We denote the symmetry group of S by $\mathcal{G}(S)$. For most sets S $\mathcal{G}(S)$ contains just one symmetry, the identity $t_{[0]}$. We say that such a set is *asymmetric* . A half-open interval is asymmetric. However, in an earlier exercise you showed that a closed or an open interval is not asymmetric; although the symmetry group of such an interval is quite simple consisting of just two congruences, the identity and the reflection about the midpoint of the interval.

EXERCISE 2.21. *Describe the symmetry groups of the following sets.*

(i) $\mathbb{Z}$*, the set of all integers.*
(ii) $\mathbb{Z}^+$*, the set of all positive integers.*
(iii) O*, the set of all odd integers.*
(iv) $I = \cup_{p\in O}[p, p+1)$.
(v) $\mathbb{Q}^+$*, the set of all positive rational numbers.*
(vi) $\mathbb{Q}$*, the set of all rational numbers.*
(vii) T*, the set of all integer powers of 2, i.e. all numbers of the form* 2^z *where* $z \in \mathbb{Z}$*.*

(viii) *D, the set of all numbers of the form* $\frac{i}{2^j}$, *where* $i, j \in \mathbb{Z}$.

CHAPTER 3

Some Applications of Real Linear Functions

3.1. Introduction

The mathematics of personal finance is a very important topic for everyone. As a topic in mathematics it is usually deferred until students study exponential functions and have the ability to sum geometric series. However, this subject can be developed very naturally through an investigation of the iteration of a linear function. That is the approach we take here.

We start by considering two specific examples. Suppose that you start a savings program: you have \$250 of your monthly pay check automatically deposited in a savings account. And let's say that the account pays 4.5% interest. How much will you have in a year? in 3 years? We can start by computing each month's balance in order. Assume that the interest is *compounded monthly,* that is, each month the interest you have earned that month is computed and added to the account. Since 4.5% is the annual interest rate, we must first compute the monthly rate which is simply $\frac{4.5\%}{12} = .375\%$.

Now let x_i denote the balance in the account at the end of month i, just after the month's interest has been added and the monthly payment is made. Since the first payment is made at the end of the first month, $x_1 = 250$ and it is convenient to include $x_0 = 0$. To compute x_2, we compute the interest earned in the second month, $\$250 \times 0.00375$, \$.9375 or 93.75 cents, and deposit it in the account to get \$250.9375 and then deposit the second months payment to get a total of \$500.9375 in the account at the end of the second month. Of course the bank would probably round this to \$500.94. However, the effects of rounding will be insignificant and can be ignored. So we will not round and we will also not carry the dollar sign in computing further balances, rounding and adding \$ only at the end. Thus,

$$x_2 = 250.9375 + 250 = 500.9375.$$

In general, to compute x_i, we compute the month's interest on the previous balance, x_{i-1}, add it to x_{i-1} and add in the next payment of 250. There is one useful short cut that we will use: instead of computing the interest on x_{i-1}, $0.00375x_{i-1}$ and adding it to x_{i-1}, we will simply multiply x_{i-1} by 1.00375. So

$$x_i = 1.00375x_{i-1} + 250, \text{ for } i\text{= }\mathit{1,2,3}\ldots.$$

If we let $f(x)$ denote the linear function $1.00375x + 250$, $x_1 = f(x_0)$, $x_2 = f(x_1) = f(f(x_0))$, $x_3 = f(x_2) = f(f(x_1)) = f(f(f(x_0)))$, and so on. We use f^k to denote k iterations of f, we write succinctly

$$x_i = f^i(x_0).$$

We may analyze loans in a similar way. Suppose that you borrow \$15,000 at 6% to be paid off with equal monthly payments of \$290 each for 5 years. Here let x_i denote the balance still due at the end of i months. Clearly, x_0 the amount you owe at the end of the 0th month or the start of the first month is simply the amount that you borrowed, 15000. At the end of the first month, but before you make your payment the interest is added to get $1.005 \times 15000 = 15075$. Your payment is then subtracted to get your new balance $x_1 = 14785$. In general,

$$x_i = 1.005x_{i-1} - 290, \text{ for } i\text{= }\mathit{1,2,3}\ldots$$

and

$$x_i = g^i(x_0), \text{ where } g \text{ is the linear function } 1.005x - 290$$

An obvious question to ask about such a loan is: If you borrow \$15,000 for 5 years at 6%, how do you know that the monthly payments should be \$290? In other words how should you choose the constant b for $f(x) = 1.005x - b$ so that $f^{60}(15000) = 0$? Clearly, a careful investigation of iterating a linear function will be needed to answer this question.

3.2. Iterating Linear Functions

Periodic savings and loans are just two of a large body of useful applications involving iterating a linear function. And these applications have developed a special terminology. The equation $x_n = ax_{n-1} + b$ is called a *linear difference equation* and the resulting sequence of numbers $\{x_n, \text{ for } n = 0, 1, 2, \ldots\}$, is the solution to that linear difference equation with initial value x_0. As we have observed, the solution to the linear difference equation, with initial value x_0, is simply the sequence of iterates, $x_n = f^n(x_0)$, of the linear equation $f(x) = ax + b$.

Any linear function can be thought of as a linear difference equation. Let's consider, from this point of view, some specific linear functions. First, consider the translation $t_{[1]}$. If we take x_0 to be 0, then $x_1 = 1$, $x_2 = 2$, and one easily sees that $x_n = n$ for all positive integers n.

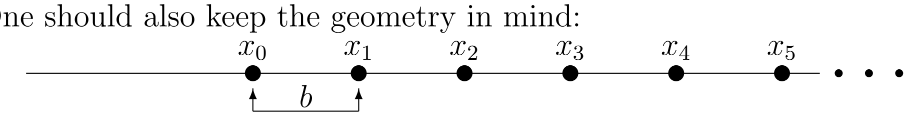
In fact it is easy to see that the difference equations associated with translations always have such simple solutions. Consider the linear difference equation corresponding to $t_{[b]}$, i.e. $x_n = x_{n-1} + b$. One computes directly that:

$$x_1 = x_0 + b,$$

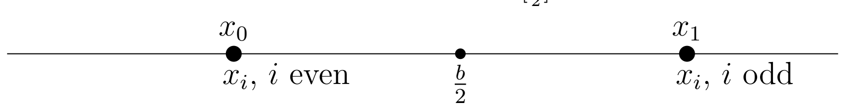
$$x_2 = x_1 + b = x_0 + 2b,$$

$$x_3 = x_2 + b = x_0 + 3b, etc.$$

One should also keep the geometry in mind:

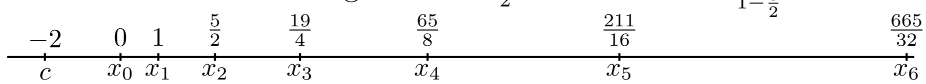


We conclude that the solution to a linear difference equation corresponding to the translation $t_{[b]}$, $x_n = x_{n-1} + b$, with initial value x_0, is simply $x_n = x_0 + nb$ or the *arithmetic progression* starting at x_0 with difference b. Conversely, each arithmetic progression is the solution to a linear difference equation corresponding to a translation. If the solution sets to linear difference equations associated with translations are simple, the solution sets to linear difference equations associated with reflections are even simpler. The solution to $x_n = x_{n-1} + b$ can be easily understood geometrically. The linear equation associated with this difference equation is the reflection $r_{[\frac{b}{2}]}(x)$, which we picture:



Thus, in the case that the associated linear function is a congruence, the solutions to a linear difference equation are rather easy to understand. They behave quite differently from the solutions to linear difference equations associated with all other linear functions. Hence, translations and reflections are usually excluded from consideration. Specifically, only linear functions of the form $x_{n+1} = ax_n + b$, where a is not equal to 0, 1, or -1 are considered. We will make these exclusions from now on. Thus, linear difference equations fall into four distinct categories: those with $a \in (-\infty, -1)$, those with $a \in (-1, 0)$, those with $a \in (0, 1)$ and those with $a \in (1, \infty)$. We will soon see that this is a good way of classifying linear difference equations in that the solutions to the linear difference equations within each category share certain distinctive properties. As it turns out, all of our applications will have a positive. So we will concentrate on the two classes with $a > 0$. Also, before we completely put the linear difference equations associated with reflections out of our mind, we should make the key observation that the center of the reflection is "central" to understanding the geometry of the solution to the linear difference equation. We illustrate this fact with a few examples.

Consider the linear difference equation $x_n = \frac{3}{2}x_{n-1} + 1$ with $x_0 = 0$. We have plotted the first few x_i, below. The linear function $f(x) = \frac{3}{2}x + 1$ is a dilation with magnification $\frac{3}{2}$ and center $c = \frac{1}{1-\frac{3}{2}} = -2$.



Since f is a dilation with magnification $\frac{3}{2}$, f moves points away from the center and successive applications of f move points away at faster rates: one application of f stretches all lengths by $\frac{3}{2}$; two applications of f stretches all lengths by $(\frac{3}{2})^2 = \frac{9}{4}$. In particular, for any two points $x < y$, $f(y) - f(x) = \frac{2}{3}(y - x)$. Using this fact and the fact that $f(c) = c$. we have:

$(x_1 - c) = (f(x_0) - f(c)) = \frac{3}{2}(x_0 - c) = \frac{3}{2} \times 2 = 3;$
$(x_2 - c) = (f(x_1) - f(c)) = \frac{3}{2}(x_1 - c) = \frac{3}{2} \times 3 = \frac{9}{2};$
$(x_3 - c) = (f(x_2) - f(c)) = \frac{3}{2}(x_2 - c) = \frac{3}{2} \times \frac{9}{2} = \frac{27}{4}.$

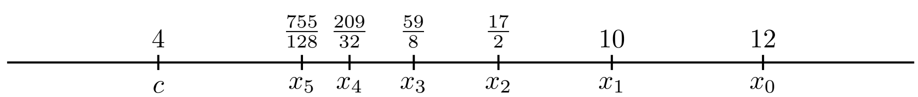
In general,

$(x_i - c) = \frac{3}{2}(x_{i-1} - c) = (\frac{3}{2})^2(x_{i-2} - c) = \cdots = (\frac{3}{2})^i(x_0 - c)$

and so $x_i = (\frac{3}{2})^i(x_0 - c) + c$. For example,

$x_4 = (\frac{3}{2})^4(x_0 - c) + c = \frac{81}{16}(3) - 2 = \frac{243}{16} - \frac{232}{16} = \frac{211}{16}.$

Now let $x_n = \frac{3}{4}x_{n-1} + 1$ with $x_0 = 12$. Again we have plotted the first few x_i, below. The linear function $f(x) = \frac{3}{4}x + 1$ is a dilation with magnification $\frac{3}{4}$ and center $c = \frac{1}{1-\frac{3}{4}} = 4$.



Since f is a dilation with magnification $\frac{3}{4}$, f shrinks distances and moves points toward the center: one application of f shrinks all lengths by $\frac{3}{4}$; two applications of f shrinks all lengths by $(\frac{3}{4})^2 = \frac{9}{16}$.

In particular, for any two points $x < y$, $f(y) - f(x) = \frac{3}{4}(y - x)$. Using this fact and the fact that $f(c) = c$. we have:

$(x_1 - c) = (f(x_0) - f(c)) = \frac{3}{4}(x_0 - c) = \frac{3}{r} \times 8 = 6;$
$(x_2 - c) = (f(x_1) - f(c)) = \frac{3}{4}(x_1 - c) = \frac{3}{4} \times 6 = \frac{9}{2};$
$(x_3 - c) = (f(x_2) - f(c)) = \frac{3}{4}(x_2 - c) = \frac{3}{4} \times \frac{9}{2} = \frac{27}{8}.$

In general,

$(x_i - c) = \frac{3}{4}(x_{i-1} - c) = (\frac{3}{4})^2(x_{i-2} - c) = \cdots = (\frac{3}{4})^i(x_0 - c)$

and so $x_i = (\frac{3}{4})^i(x_0 - c) + c$. For example,

$x_4 = (\frac{3}{4})^4(x_0 - c) + c = \frac{81}{256}(8) + 4 = \frac{81}{32} + \frac{128}{32} = \frac{209}{32}.$

EXERCISE 3.1. *For each of the liner difference equations* $x_n = -\frac{3}{2}x_{n-1} + 1$ *with* $x_0 = 0$ *and* $x_n = -\frac{3}{4}x_{n-1} + 1$ *with* $x_0 = 0$

(i) *graph the first few terms;*
(ii) *compute the center* c*;*
(iii) *compute* $x_i - c$ *for the first few values of* i*.*

Now we repeat these calculation for an arbitrary linear difference equation $x_n = ax_{n-1} + b$ with center $c = \frac{b}{1-a}$. Write f in slope.-center form, $f(x) = ax + (1-a)c$, and rewrite the difference equation in this form $x_n = ax_{n-1} + (1-a)c$. This gives $x_n - c = a(x_{n-1} - c)$; from which, we conclude that $x_n - c = a^n(x_0 - c)$. We have proved:

THEOREM 4. *Let* $x_0,\ x_1,\ ...,x_n,\ ...$ *be a solution to the linear difference equation* $x_n = ax_{n-1} + b$ *(where* $a \neq -1, 0, 1$*). Let* $c = \frac{b}{1-a}$*. Then:*

$$x_n = a^n(x_0 - c) + c$$

COROLLARY 4.1. *Let* $x_0,\ x_1,\ ...,x_n,\ ...$ *be a solution to the linear difference equation* $x_n = ax_{n-1} + b$ *(where* $a \neq -1, 0, 1$*).*

(i) *If* $0 < a < 1$*, then the solution is a monotone sequence of points starting at* x_0 *and converging toward* $\frac{b}{1-a}$*.*
(ii) *If* $1 < a$*, then the solution is a monotone sequence of points starting at* x_0 *and diverging from* $\frac{b}{1-a}$*.*
(iii) *If* $-1 < a < 0$*, then the solution is a sequence of points starting at* x_0 *and converging toward* $\frac{b}{1-a}$*, but alternating from one side of* $\frac{b}{1-a}$ *to the other.*
(iv) *If* $a < -1$*, then the solution is a sequence of points starting at* x_0 *and diverging from* $\frac{b}{1-a}$*, but alternating about* $\frac{b}{1-a}$*.*

EXERCISE 3.2. *For each of the following linear difference equations with each of the given initial values, (1) find the fixed point* c*, (2) write out the formula for* x_n*, (3) describe the solution sequence as monotone or alternating and converging or diverging, (4) graph the first several terms of the solution.*

(i) $x_n = \frac{7}{5}x_{n-1} - 1$ *for* $x_0 = 0, 5, 10$
(ii) $x_n = -\frac{7}{5}x_{n-1}1$ *for* $x_0 = 0, \frac{1}{2}, 2$
(iii) $x_n = \frac{5}{7}x_{n-1} - 1$ *for* $x_0 = 0, 5, 10$
(iv) $x_n = -\frac{5}{7}x_{n-1} - 1$ *for* $x_0 = 0, \frac{1}{2}, 2$

Given any sequence of real numbers $x_0,\ x_1,\ \cdots,\ x_n,\ \cdots$, one may ask if it is the solution set of a linear difference equation.

EXERCISE 3.3. *Prove that* $x_0,\ x_1,\ \cdots,\ x_n,\ \cdots$ *is the solution set of a linear difference equation if and only if, for each integer* $i \geq 0$*, we* *have:*

$$(x_{i+3} - x_{i+2})(x_{i+1} - x_i) = (x_{i+2} - x_{i+1})^2.$$

We may now consider a wide variety of applications of linear difference equations.

3.3. Loans and Savings programs

Recall our first example from the introduction: you have \$250 of your monthly pay check automatically deposited in a savings account that pays 4.5% interest. We calculated the monthly interest rate to be $\frac{4.5\%}{12} = .375\%$ and showed that x_i, the balance in the account at the end of the $i$the month, for $i = 1, 2, \ldots$, is given by the linear difference equation

$$x_i = 1.00375x_{i-1} + 250, \text{ for } i\text{= }1,2,3\ldots,$$

with $x_0 = 0$. To solve this we first compute $c = \frac{250}{1-1.00375} = -\frac{200000}{3}$. Hence, by the theorem we have just proved,

$$x_i = (1.00375)^i(\frac{200000}{3}) - \frac{200000}{3}.$$

So, for example, at the end of 3 years you will have

$$(1.00375)^{36}(\frac{200000}{3}) - \frac{200000}{3} = 9616.52$$

dollars in the account (rounded to the nearest cent).

We can use this formula in a variety of ways. Suppose that you decide that \$9616.52 won't be enough for your needs and that you really need \$12,000 at the end of 3 years. How big should your monthly payments be? We have $c = \frac{b}{1-a}$ and $x_{36} = 1.00375^{36}(0 - c) + c$. Setting $x_{36} = 12000$ and solving for c gives $c = \frac{12000}{1-1.00375^{36}}$. So $b = (1 - a)c = \frac{.00375 \times 12000}{1.00375^{36}-1}$, which computes out at \$311.96.

Exercise 3.4. *Verify that saving \$312 monthly in an account paying 4.5% will yield just a little over \$12,000 in three years. How much of the \$12,000 is interest that your investment earned?*

Now let's develop the general formula for regular savings. Suppose that you deposit b dollars at the end of each period, for n periods at a periodic rate of i. (The periodic rate is simply the annual rate divided by the number of periods per year.) Letting x_n denote the value of the investment at the end of the n^{th} period, the linear difference equation which describes this is:

$$x_n = (1 + i)x_{n-1} + b, \quad \text{where} \quad x_0 = 0.$$

We easily compute c to be $c = \frac{b}{(1-(1+i))} = -\frac{b}{i}$; so the value of x_n is:

$$x_n = (1+i)^n(0 - (-\frac{b}{i})) + (-\frac{b}{i}) = (1+i)^n(\frac{b}{i}) - \frac{b}{i} = \frac{(1+i)^n - 1}{i}b.$$

We have proved:

THEOREM 5. *The (future) value F of n regular periodic payments of \$$b$ at a periodic rate of i is given by:*

$$F = \frac{(1+i)^n - 1}{i}b.$$

Financial calculators and the financial packages for scientific calculators have this formula in memory. In Texas Instrument calculators the formula is stored in the following form:

$$FV + \frac{(1 + \frac{I}{P/Y})^N}{\frac{I}{P/Y}}PMT = 0,$$

where FV is the future value, I is the annual rate, N is the total number of periods or payments, P/Y the number periods or payments per year and PMT is the amount of each payment. The user enters any four of the five parameters and the calculator will solve the equation for the fifth. If you enter $I = 4.5$, the annual rate; $P/Y = 12$, monthly payments; $N = 36$, for 36 months; $PMT = 250$, payments of \$250 and press SOLVE, the calculator will report $FV = -9616.52$. The minus sign is the result of the way that the formula is stored. But there is a nice way to interpret the sign. Think of the money as moving between you and the bank. If we choose to think of money moving from you to the bank as moving in the positive direction, then money moving from the bank to you is moving in the negative direction. So if you had entered $PMT = -250$ the calculator will report $FV = 9616.52$.

Actually, the calculator has one additional variable PV or present value. Consider opening the account with an initial deposit in the amount of PV, not necessarily the same size as the later regular payments. Thinking about this for just a moment, we realize that PV is simply x_0. Redoing the calculations that preceded the statement of Theorem **??** with x_0 not set equal to 0:

$$x_n = (1+i)^n(x_0 - (-\frac{b}{i})) + (-\frac{b}{i}) = (1+i)^n x_0 + \frac{(1+i)^n - 1}{i}b.$$

Substituting the "calculator notation" for the variables while changing the sign of FV and moving it to the "other side" gives:

THE CALCULATOR FORMULA

$$FV + (1 + \frac{I}{P/Y})^N PV + \left((1 + \frac{I}{P/Y})^N - 1\right) \frac{PMT}{\frac{I}{P/Y}} = 0.$$

This one-size-fits-all formula has many different applications. One of the simplest is simply tracking an investment. Suppose you invest \$10,000 at 6% but make no further deposits. What is the value of the investment in N months, assuming it is compounded monthly? Here $PMT = 0$ and we simply have:

$$FV + (1 + \frac{I}{P/Y})^N PV = 0$$

or, in our case, $FV + (1.005)^N 10000 = 0$. At the end of five years the investment has grown to \$12,833.57 ($FV = -12,833.57$ with the calculator sign convention).

This simple $PV - FV$ formula has another important use. Suppose that I offer to sell you a *bond* that will pay $13,000$ in 5 years, exactly 60 months from now. How much is it worth now? In view of our previous computation and assuming that the generally available interest rate for investments is 6%, \$10,000 would seem like a good price. We could ask the exact amount that must be invested today at an annual rate of 6% compounded monthly to yield \$13,000 at the end of 5 years. We call that the *present value* of this \$13,000 bond. This bond has a present value of \$9,637.84.

$PV = -FV(1 + \frac{I}{P/Y})^{-N} = 13000(1.005)^{-60} = -9637.84$

Three comments about present value. First of all, it is very useful in computing things like total assets that may include money that you have now along with money that you will get at some future date. Second, the present value depends on the assumed investment rate and would come out differently under different assumptions. Third, it does not take into account *risk*: it is based on the assumption that the bond will certainly pay its face value in 5 years and that the bank will pay 6% for the next 5 years. In spite of these concerns, present value is an extremely useful concept.

EXERCISE 3.5. *Upon entering the work force at age 20, Sue starts saving for her retirement at age 65. She open an account at an annual rate of 5.6% compounded monthly into which she will make regular monthly payments.*

(i) *If her payments are \$100, how much will be in the account at the time of her retirement? How much of that amount represents interest earned?*

(ii) *How big should her payments be if she wishes to retire a millionaire?*

EXERCISE 3.6. *A manufacturer has just bought a piece of equipment for \$500,000. He expects to have to replace this piece of equipment in ten years and wishes to set up a "sinking fund" to pay for the replacement, i.e., an annuity which, in ten years, will be worth the cost of the replacement.*

(i) *What should be the size of his monthly payment into the sinking fund? In answering this question, use the following data: the manufacturer assumes that the cost of the replacement will go up by an annual inflation rate of 3% and that the investments in the annuity will grow at an annual rate of 4.5% compounded monthly. First compute the amount that the manufacturer assumes that he will need in ten years and then compute the payments that will achieve that target.*

(ii) *After five years the economic climate has changed and our manufacturer decides to adjust is payments into the sinking fund. He notes first that, at this time, the cost of a replacement is \$600,000, not the \$579,637 predicted by a 3% inflation rate and that an inflation rate of 4% is expected for the next five years. The good news is that the funds now in the sinking fund along with all future payments will earn 6% compounded monthly. Find his new payments.*

EXERCISE 3.7. *Your rich aunt offers you choice of*

(i) \$2,500 today *(ii) \$3000 in 3 years* *(iii) \$3400 in 5 years.*

You suspect that she is testing your ability to make wise financial decisions. Which option would you choose if the interest rate is 6%? For each option answer the question "is there an interest rate that make that option the best choice?"

Now let's return to the second type of calculations discussed in the introduction those involved with loans. Consider a loan with payments of b dollars and charging interest at i% per period. If we denote the amount due on the loan at the end of the k^{th} period, the linear difference equation which describes a loan is:

$$x_k = (1 + i)x_{k-1} - r,$$

where x_0 is the amount of the loan. We easily compute c to be $\frac{-r}{(1-(1+i))} = \frac{r}{i}$; so the value of x_k is:

$$x_k = (1 + i)^k(x_0 - \frac{r}{i}) + \frac{r}{i}.$$

Let n denote the length of the loan, i.e.,the number of periods or payments needed to pay off the loan. Then $x_n = 0$. Setting $x_0 = L$ and collecting the terms involving $\frac{r}{i}$, gives the formula in Part (iii) below. Then setting $k = n$ and $x_n = 0$ and solving for L, gives the formula in Part (i). Finally solving this equation for r, gives the formula in Part (ii).

THEOREM 6.

(i) *If L denotes the amount of a loan in dollars to be paid off in n periodic payments of r dollars at a periodic interest rate i, then:*

$$L = \frac{(1+i)^n - 1}{i(1+i)^n} r$$

.

(ii) *If you borrow L dollars for n periods at a periodic rate i, your payments will be given by:*

$$r = \frac{i(1+i)^n}{(1+i)^n - 1} L$$

.

(iii) *If you borrow L dollars with periodic payments of r dollars at a periodic rate i, then, x_k, the amount due on the loan after k periods is given by:*

$$x_k = (1+i)^k L - \frac{(1+i)^k - 1}{i} r$$

.

EXERCISE 3.8. *Write out the proof of Theorem* **??**.

EXERCISE 3.9. (i) *Substitute the value for L given in part (i) of the theorem into the formula for x_k in part (ii). Then simplify to get:*

$$x_k = \frac{(1+i)^{n-k} - 1}{i(1+i)^{n-k}} r$$

.

(ii) *Interpret x_k as the amount that one could borrow for $n - k$ periods at a periodic rate of i with payment size r proving the following corollary.*

COROLLARY 6.1. *The balance still due on a loan after k periods is equal to the amount you could borrow for $n - k$ periods at a periodic rate of i and payment size r where i, r and n are the periodic rate, payment and length of the original loan.*

How do loans fit into our financial calculator formula? Recall the formula:

$$FV + (1 + \frac{I}{P/Y})^N PV + \left((1 + \frac{I}{P/Y})^N - 1\right) \frac{PMT}{\frac{I}{P/Y}} = 0.$$

and compare it to the formula below obtained by multiplying both sides of formula (i) of Theorem **??** by $(1+i)^n$

$$(1+i)^n L = ((1+i)^n - 1)\frac{r}{i}$$

In the calculator notation $n = N$, $i = \frac{I}{p/y}$ and $r = PMT$; rewriting our equation in this form we have:

$$(1 + \frac{I}{P/Y})^N L = \left((1 + \frac{I}{P/Y})^N - 1\right) \frac{PMT}{\frac{I}{P/Y}}.$$

Recalling the calculator convention we move $(1 + \frac{I}{P/Y})^N L$ to the other side without changing its sign and see that L corresponds to the present value PV. *A loan is simply the present value of the payments.*

Exercise 3.10. *Sue wishes to buy a car, she has \$2,500 for a down payment and the "going" interest rate is 10.5%.*

(i) *How big will her payments be if she wishes to buy a \$13,500 car and pay it off in 4 years? How much in interest will she pay over the life of the loan?*
(ii) *If she can make monthly payments of only \$200, how expensive a car can she buy with a 4 year loan, a 5 year loan? In each case, how much interest will she pay over the life of the loan?*

The difference equation approach to loans allows us to get far more information about the loan than just the size of the payments. Recall, that the linear difference equation is for the balance due on a loan after k payments is:

$$x_k = (1+i)x_{k-1} - r$$

Recall also that the fixed point for this difference equation is $\frac{r}{i}$. Since $a = (1+i) > 1$, the terms x_k diverge monotonically from $\frac{r}{i}$. If the value of the loan, x_0, is less than $\frac{r}{i}$, then successive values will get smaller and eventually the loan will be paid off; if on the other hand x_0 is larger than $\frac{r}{i}$, successive values will get larger and the loan will never be paid off! The condition insuring the eventual payment of the loan, $L = x_0 < \frac{r}{i}$, gives $iL < r$. The product iL dollars is the interest on L dollars for one period. Hence, if $r > iL$, the first payment of r dollars will pay the accrued interest and will reduce the amount owed by $r - iL$

dollars. Since the new debt is smaller than L, repeated payments will eventually pay off the loan. But, if $r < iL$, the first payment will not even cover the accrued interest and the debt will increase. Clearly, in this case, the debt increases at each payment and the loan is never paid off.

Consider a \$10,000 five year loan at 12%. Using the formula in Theorem **??**b, we have that the monthly payments will be \$222.44. which is more than double the interest iL or \$100. Now compare this with a 30 year \$100,000 mortgage also at 12%. In this second case, iL is \$1,000 while the payments compute to only \$1028.62! This is the major difference between short-term and long-term loans: A large part of all payments of a short-term go toward reducing the debt while the early payments on a long-term loan are almost all interest. Of the first payment of the mortgage just described, \$1,000 went toward interest while the debt was reduced by only \$28.62.

To better understand this difference between short term and long term loans, let us consider two specific examples: a \$100,000, 25-year mortgage at 6% and a \$100,000, 5 year business loan also at 6%. The monthly payments for the mortgage are:

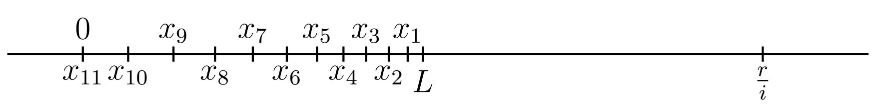
$$r = \frac{.005(1.005)^{300}}{1-(1.005)^{300}} = 644.30.$$

while the monthly payments on the 5 year business loan are:

$$r = \frac{.005(1.005)^{60}}{1-(1.005)^{60}} = 1933.28.$$

Since the interest on \$100,000 for one month at 6% is \$500, we see that only \$144.30 of the first mortgage payment goes to reducing the debt while the first business loan payment reduces the debt by \$1433.28. Just why this dramatic difference occurs can best be understood using our one-line graphs.

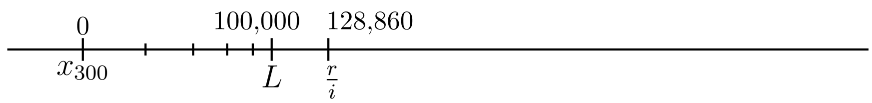
Consider an arbitrary loan in the amount L for n periods with a periodic rate of i and payments of r dollars per period. In the following figure below we have indicated the center $(\frac{r}{i})$ of the difference equation governing this loan. For this illustration we have taken $n = 11$.



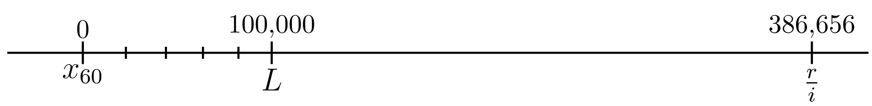
For a loan, the magnification of the associated similarity is $(1 + i)$. Thus the distance between the center and x_1 is $(1+i)$ times the distance between the center and L or the distance between x_1 and L is i times the distance between the center and L:

$$(1+i)(\frac{r}{i}-L)=(\frac{r}{i}-x_1) \quad \text{gives} \quad i(\frac{r}{i}-L)=(L-x_1) \quad \text{(check this)}.$$

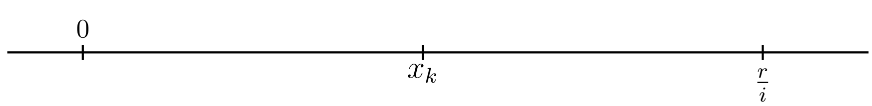
This helps to explains the fundamental difference between a long term and a short term loans. In a long term loan such as the mortgage described above, the value of the loan is relatively close to the center and the balance due does not decrease very much at the outset. For this mortgage, we have $\frac{r}{i}=\frac{644.30}{.005}=128,860$ and some values for the balance due on this mortgage are sketched below. Of the first payment, \$500 goes for interest while the balance due is reduced by only \$144.30. The value of x_1 (\$99,855.70) is so close to x_0 that it cannot be distinguished from x_0 on our graph. We have graphed the balance due in 5-year intervals: $x_{60}=\$89,932.20$, $x_{120}=\$76,352.21$, $x_{180}=\$58,034.86$ and $x_{240}=\$33,327.50$.



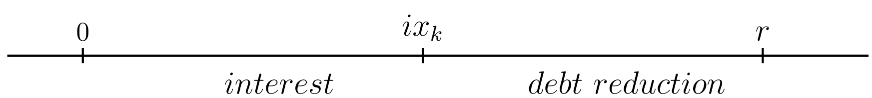
For the 5 year business loan we have $\frac{r}{i}=\frac{1933.23}{.005}=386,656$. Of the first payment, \$500 goes for interest while the debt is reduced by \$1433.23. So debt reduction is more significant from the very start and the balance due values do not bunch up as much about x_0. We have graphed the balance due each year: $x_{12}=82,319.69$, $x_{24}=63,548.84$, $x_{36}=43,620.35$ and $x_{48}=22,462.66$.



This method of picturing a loan has one other useful feature. If we make a simple change of scale, we have a geometric representation of the distribution of each payment between interest and debt reduction:



Multiplying through by i, we have:



Of the $(k+1)$st payment of r dollars, ix_k dollars is the interest on the previous balance while the debt is reduced by $r - ix_k$ dollars.

EXERCISE 3.11. *Upon entering the work force, Sue starts saving for a car for a her \$15,000 dream car. She open an account at an annual rate of 4.6% compounded monthly into which she will make regular monthly payments. She also has the option of borrowing at a 10.2% annual rate.*

(i) *How big should her payments be if she wishes to have the \$15,000 for the car in 3 years?*
(ii) *If she borrows the full \$15,000 to buy the car now and pays off the loan in 3 years, how big will her payments be?*
(iii) *Sue decides to save for 18 months, then buy the car and pay off the rest with a loan over the next 18 months. She wants her payments into the savings account and her loan payments to be about the same. What will the payments be?*

EXERCISE 3.12. *Ten years ago the Smiths bought a house taking out a 25 year \$75,000 mortgage at 8.6%.*

(i) *How big are their payments.*
(ii) *Now, after exactly 10 years, how much do they still owe?*
(iii) *For a flat fee of \$1,000 (which can be added to the loan) they can refinance the loan at 6.8%, resulting a 15 year mortgage for the balance due plus \$1,000. If they were to do this what would their new payments be?*

Loans can be configured differently. For example, with a credit card or a home equity loan each payment consists of the total interest accrued and a fixed percentage of amount of the loan or of the amount still due on the loan. In this case, the early payments will be lager than in a "traditional" loan and will then decrease each month. Also, the total interest paid will be less than with a traditional loan, assuming the same rate is charged for both loans. To be specific let us consider the case of the Johnsons.

EXERCISE 3.13. *The Johnsons paid off their home mortgage 5 years ago. Now they plan to buy a new car and to pay for it they take a home equity loan of \$15,000 on their house at 9.5% to be paid off in monthly payments over 5 years. For this, each monthly payment is equal to* $\frac{1}{60}$ *of the amount of the loan plus the interest on the unpaid balance.*

(i) *How much is their first payment?, their second payment?, their third payment?, their last payment?*
(ii) *What is the total interest that they will pay?*

(iii) *If they could get a standard 5 year car loan at the same rate, what would their payments be?*
(iv) *What is the total interest paid with the standard loan?*
(v) *Explain the difference in the amounts of interest paid by the two schemes.*

EXERCISE 3.14. *Consider loans of the type just described: the payment p_n due at the end of the nth time period is given by $p_n = \frac{L}{N} + ix_n$, where $L = x_0$ is the size of the loan, i is the periodic rate, N is the number of periods and x_n is the amount due at the end of the nth period (after the payment is made).*

(i) *Show that the balance due x_n is given by the "linear difference equation" $x_n = x_{n-1} - \frac{L}{N}$. [This is not really a linear difference equation by our definition since $a = 1$.]*
(ii) *Find the "linear difference equation" for p_n.*
(iii) *Find a formula for x_n that does not involve any other x_k or p_k.*
(iv) *Find a formula for p_n that does not involve any other p_k or x_k.*

EXERCISE 3.15 (Research Project). *Before high speed calculators, the balance due on a loan was often computed by a rather simple computation called the "Rule of 78's". Look up this method, explain it and then discuss just how accurate it is.*

3.4. Population Models

Consider the following problem: you own a piece of land containing a pond which you would like to stock with trout. Trout will not reproduce in this pond but roughly two-thirds of the fish population will survive from one year to the next. Suppose that you initially stock the pond with 5000 fish and then add 2000 in each successive year. At what level will the fish population stabilize? One easily sees that the population levels in your pond are given by a linear difference equation: if p_n denotes the size of the fish population in the nth year, just after the new fish have been added, then the p_n are the terms in the solution to the linear difference equation $p_n = \frac{2}{3}p_{n-1} + 2000$ with initial condition $p_0 = 5000$. The fixed point for this difference equation is $c = \frac{2000}{1-\frac{2}{3}} = 6000$. So, in a few years the fish population should stabilize at about 6000. Note that, contrary to our intuition, this stable value does not depend on the size of the initial stocking. However, the size of the initial stocking will effect the length of time that it takes the population to stabilize. These observations are illustrated in the following exercises.

EXERCISE 3.16. *Consider the pond described above. We say that the population has stabilized once it is within 100 of the fixed value.*

(i) *In the above problem, how many years will it take for the fish population to stabilize?*
(ii) *If the pond is initially stocked with 10,000 fish, how many years would it take for the population to stabilize?*
(iii) *What should be the size of the initial stocking if you wish the population to stabilize as soon as possible? Formulate your answer as a general principle.*
(iv) *If you wish the stable population in your pond to be 10,000, how many fish should be added each year?*

EXERCISE 3.17. *All of the fish that you have been using to stock your pond come from a friend who owns a pond in which trout do reproduce. Each year the trout population in his pond increases by 25% until the maximum population of 12,000 trout is reached. So you remove 5000 fish from his pond the first year and 2000 a year there after. When you initially stocked your pond his pond was at its capacity of 12,000. If we let q_n denote the number of trout in your neighbor's pond, we have then that $q_0 = 7,000$.*

(i) *Give the linear difference equation that governs the population size in your friend's pond.*
(ii) *Assume that you you continue to stock your pond with 2000 trout each year from your friend's pond. What will the trout population be in your friend's pond after 10 years?*
(iii) *How could this have been avoided?*

There are innumerable variations on this "population model". The population to be modeled could be fish or some other animals or people. In such models, the subscript n denotes the number of time periods: for people and larger animals n often stands for the number of years or decades which have elapsed, for smaller animals (mice, for example) n could stand for the number of months, days may be appropriate for insects such as roaches and hours for bacteria. The magnification a is usually greater than 1 and $100 \times (a - 1)$ represents the percentage population increase (births minus deaths divided by population) for the given time period due to the natural biological processes. The constant b then represents the change in population due to immigration and emigration (immigration minus emigration, to be precise). Whenever $b \geq 0$ successive populations x_n will grow unbounded indicating that the population will eventually out grow its environment. At this point, our simple linear difference equation model becomes invalid. For such models, the relevant question is: how long will it take the population to reach the environmental limits? Another question often asked of such models is: how long will it take for the population to double? In computing the formula for the nth term of a solution to a difference equation, exponential functions arise (in general, unconstrained population growth is exponential). It is not surprising then that, in solving the questions just posed, logarithmic functions will arise.

EXERCISE 3.18. *Every 10 years the government takes a population census and publish the results. Look up these data and set up a population model* $p_n = ap_{n-1} + b$ *for the United States. Start your model in 1900 (*p_0 *is the population at 1900) and let* n *denote the number of lapsed decades. Then* p_n *is the population in 19*n*0 for* $n = 0, \cdots, 9$*, and predicted populations for the years 1900 + 10*n *when* $n \geq 10$*. As stated above,* $100 \times (a-1)$ *represents the percentage population increase (births minus deaths) for the given time period due to the* natural biological processes - *be careful here, your data may include the immigration/emigration figures. Once you have constructed your model answer the following questions.*

(i) *What size population does your model predict for 2010?, 2020?*
(ii) *When does your model predict that the population of the US will reach 300,000,000?*
(iii) *How long will it take for the 1990 population to double?*

EXERCISE 3.19. *Consider the population model* $p_n = ap_{n-1} + b$

(i) *Find the formula for the number of time periods needed for the population to double assuming that* $b = 0$
(ii) *Find the formula for the number of time periods needed for the population to double assuming that* $b \neq 0$

3.5. Radioactive Decay

Certain physical processes may also be described by linear difference equations. One classic example is radioactive decay. Given a sample of a radioactive substance, its level of radioactivity decreases by a fixed percentage each year. For example, assume that the level of radioactivity for this substance decreases by one half of one percent each year. Thus, from one year to the next, it retains 99.5% of its radioactivity and the governing linear difference equation is $r_n = .995r_{n-1}$ where r_i is the measure of the radioactivity of the sample of the substance after i years. One question that we may ask is: how long will it take for the radioactivity of this sample to drop to one half of its original value? Using our formula for the nth term, we conclude that $r_n = (0.995)^n r_0$

and we see that r_n will equal $\frac{1}{2} \times r_0$ when $(0.995)^n = 0.5$. Taking the natural log of both sides, we get:

$$n \ln(0.995) = ln(0.5),$$

$$n = \frac{ln(0.5)}{ln(0.995)} = 138.28.$$

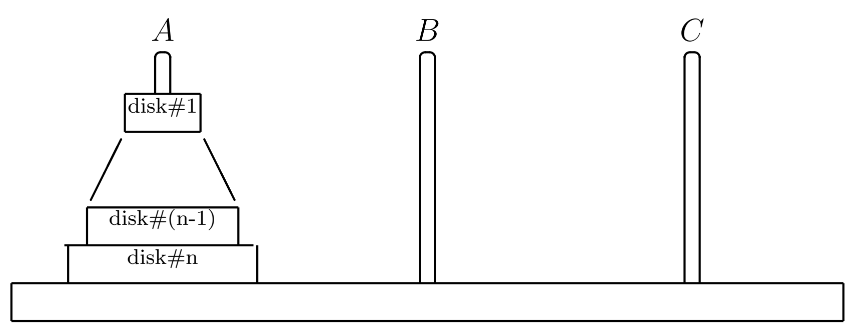
We observe that the 138.28 years does not depend on the size of the sample, the initial radiation level r_0 or by what units radiation is measured. This number, called the *half-life* of the substance.

EXERCISE 3.20. .

(i) *If a given radioactive substance has a half-life of 200 years, by what percentage will its radioactivity be reduced each year?*
(ii) *Look up the half-life for several radioactive substances and write out the difference equations which their radioactivity levels over time.*

3.6. The Towers of Hanoi

For an entirely different application, we consider the mathematical puzzle called the Towers of Hanoi. The puzzle consists of three posts and a set of n disks which have holes in their centers and slide over the posts. The disks are all of different diameters and are arranged from the largest (on the bottom) to the smallest (on the top) on post A. (See the diagram below.)



The object of the puzzle is to transfer all of the disks from post A onto post B in as few moves as possible. Moves are governed by the following rules:

1. Only one disk may be moved at a time;
2. Disks may only be moved from post to post;
3. No disk may be placed on top of a smaller disk.

If the number of disks is 1, we simply move disk#1 from post A to post B in 1 move. If there are two disks, we "park" disk#1 on post C (1 move), then move disk#2 to post B and finally move disk#1 to post B for a total of 3 moves. Furthermore it is not too hard to see that you actually need 3 moves to transfer 2 disks from post A to post B. Let m_i denote the least number of moves needed to transfer i disks from one post to another; we have $m_1 = 1$, $m_2 = 3$ and, of course, $m_0 = 0$. Now, in order to move disk#n from post A to post B, we must park disk#1 through disk#$(n-1)$ on post C. This can be done most efficiently in m_{n-1} moves. Disk#n is then move to post B. Finally, the the disks on C are transferred to post B, again in m_{n-1} moves. The total number of moves needed to transfer n disks from A to B is $m_{n-1} + 1 + m_{n-1}$ and m_n is given by the difference equation:

$$m_n = 2m_{n-1} + 1.$$

EXERCISE 3.21. *Solve this linear difference equation and give the formula for* m_n*. Verify that the formula works for* $n = 0, 1, 2, 3$ *and* 4.

The story behind the name "Towers of Hanoi" is that the transfer of 64 gold disks is actually being carried out in a Buddhist monastery near Hanoi. The monks started at the beginning of the world and the world will end when the transfer is completed.

EXERCISE 3.22. *There is a ceremony for the moving of a disk so each single move requires exactly one minute. Furthermore, according to the story, the world began exactly 12,544 years ago.*

(i) *Should we be worried about the world ending in our lifetime?*
(ii) *How long would a "24-disk world" last?*
(iii) *How many disks would a 1 billion year long world need?*

Things could be speeded up (although it is not clear that we would want to do this) if there was another parking post, 4 posts in all. Finding the best strategy in the four post problem can be challenging. While it is not of direct interest to study of the linear function, the following question is: letting f_n denote the least number of moves needed to transfer n disks from one post to another post when there are a total of 4 posts available, are the f_n the solution set of a linear difference equation?

EXERCISE 3.23. *Compute directly* f_1, f_2, f_3 *and* f_4 *and see if you can find a linear difference equation which will give these values. If you cannot, prove that it cannot be done; if you can, compute* f_5 *directly and from the difference equation and compare the results.*

3.7. Weight Control

One interesting application of linear difference equations is to weight control. Lets consider Jack. He weighs 80 kilograms and, due to a high blood pressure problem, his doctor has suggested that he reduce his weight by 10% to 72 kilograms. Jack maintains his weight of 80kg on an average of 2480 calories per day. Dividing 2480 by 80, we see that Jack needs 31 calories per kilogram to maintain his weight. This number will remain constant as Jack changes the quantity of his daily intake as long as Jack does not change the quality of his diet nor his daily level of activity. So Jack's first reaction was: "This isn't too difficult; all I have to do is to reduce my daily intake by 10% to 2232 ($.9\times 2480$) calories per day." The doctor acknowledged that Jack's calculations were correct but pointed out that, while a diet of 2232 calories per day would maintain a weight of 72kg, it would literally take years for Jack to get his weight down to that level on a diet of 2232 calories per day. In making his calculation, Jack was unaware of another important constant: the number of calories he needed to "burn off" in order to lose 1 kilogram in weight. Given Jack's life style and metabolism the doctor believes that it will take a reduction of 7000cal in intake over a period of time for Jack to lose 1kg.

The doctor and Jack agree that Jack will try to get down to the 72kg over the next 3 months (90 days). Jack, who is very good at at arithmetic, calculated $\frac{8\times 7000}{90} = 622$ and $2480 - 622 = 1858$ and concluded that, if he reduced his daily intake to 1858 calories per day, he would reach his target weight in the 90 days. While he made his computations, the doctor consulted a table and informed Jack that he must go on a 1725cal per day diet! Jack was surprised and confused. He showed the doctor his calculations and asked why they were in error. The doctor responded that the calculations were correct but the argument was wrong.

Exercise 3.24. *Explain the error in Jack's argument.*

When Jack finally understood the error in his argument, he asked the doctor where the 1725cal figure came from. The doctor showed Jack the table and explained that the entries in the table were computed from a linear difference equation! Jack went home, learned all about difference equations and finally came up with the following (correct) analysis.

Let d denote Jack's daily caloric intake, 2480 now and some other value in the future yet to be computed. Let x_0 denote Jack's weight today ($x_0 = 80kg$) and let x_n denote Jack's weight n days from now.

If all goes well, x_{90} will equal 72kg. For the nth day, we make the following computation:

$$\frac{d - 31x_{n-1}}{7000} + x_{n-1} = x_n.$$

which we explain by observing that $d - 31x_{n-1}$ is the excess (when positive) or deficit (when negative) in calories on the nth day and noting that this will result in a $\frac{d-31x_{n-1}}{7000}$ kilogram weight change (increase when positive, decrease when negative) on the nth day. Thus Jack's daily weight is given by the linear difference equation:

$$x_n = \frac{6969}{7000}x_{n-1} + \frac{d}{7000}.$$

We easily compute the fixed point to be $\frac{d}{31}$. So, if $d = 2232$ (31×72 - Jack's first solution), Jack's weight will eventually approach 72kg. But since the slope $\frac{6969}{7000} \sim 0.9956$ is so close to 1, it will indeed take several years to get within a half of a kilogram of 72kg.

EXERCISE 3.25. *If Jack were to go on a 2232cal diet what would his weight be in 1 year? in 2 years? in 3 years?*

EXERCISE 3.26. *Set up the difference equation for a diet of* 1860 *calories per day and track weight at 30 day intervals until it reaches 72kg.*

Returning the the difference equation with d unspecified, we computing the value of the nth term:

$$x_n = (\frac{6969}{7000})^n(x_0 - \frac{d}{31}) + \frac{d}{31}.$$

Substituting 80 for x_0 and solving for d gives:

$$d = \frac{31(x_n - (\frac{6969}{7000})^n 80)}{1 - (\frac{6969}{7000})^n}.$$

Now substituting 72 for x_n and 90 for n, we compute d to be 1726.9 which was rounded to 1725 in the doctor's table.

EXERCISE 3.27. *A "crash" diet is one that one that is constructed to achieve a dramatic weight loss in a short period of time. These involve drastic and dangerous cuts in caloric intake. For Jack to achieve his 10% weight reduction in 60 days how many calories a day would he be permitted? What about 30 days?*

EXERCISE 3.28. *For the typical person, there are two basic constants which we denote by c and C: c in the number of calories needed to maintain 1 kilogram of that persons weight and C the number of* *calories needed to change that persons weight by 1 kilogram. These are 31 and 7000, respectively, for Jack. Assume our typical person starts a diet with a daily caloric intake of* d *calories and let* x_i *denote the weight of this person after being on the diet for* i *days.*

(i) *Set up the linear difference equation governing this diet.*
(ii) *Find the equation for* x_n.
(iii) *Solve this equation for* d.
(iv) *Use your formula to set the diet for a 84kg person with constants* $c = 33$ *and* $C = 7200$ *who wishes to loose 5kg in 60 days.*

CHAPTER 4

Linear Functions in Business

4.1. Supply and Demand

The *demand function* for a product is usually introduced with p (the sale price of the product) as the dependent variable and q (the quantity of the product sold) as the independent variable. The demand function is always a decreasing function: price decreases as quantity increases. This may not be so obvious. But, if we consider the inverse function, quantity as a function of price, we clearly see that, as the price increases, the quantity sold decreases.

Consider the following demand function for a small appliance:

$$p = -0.0004q + 160.$$

If 100,000 appliances are produced the price will be \$120.00; if the quantity manufactured is doubled to 200,000, the price will drop to \$80.00. Like most mathematical models for "real life" problems, this model only makes sense within a certain range. For example, if a half a million appliances are produced the price will *not* fall to -\$40.00. Let us assume that our model is reasonably accurate in the production range from 50,000 to 250,000 appliances.

The *supply function* for the product is also represented with p as the dependent variable and q as the independent variable. As with the demand equation, the behavior of the supply equation is best understood when the roles of the variables are interchanged: as price increases, manufacturers will tend to produce more items. Thus, the supply function is an increasing function.

Consider the following supply function for the same small appliance:

$$p = 0.0005q + 50,$$

which we assume to be valid on the same production range. Thus, 100,000 appliances will be produced for an expected price of \$100.00 while the production level will be set at 200,000 if the expected sales price is \$150.00. If, on the other hand the price drops to \$75 per appliance, only 50,000 will be produced.

EXERCISE 4.1. *The supply equation is clearly known to the manufacturer. But, the demand equation is a construct based on experience and market research. Assume then that the above demand equation is valid but unknown to our manufacturer. At the outset our manufacturer feels that the appliance will sell well at $100.*

(i) *How many appliances will the manufacturer produce in this first month? How many of these will he be able to sell?*
(ii) *Based on his experience this first month he decides to produce 120,000 appliances in the second month. At what price will he sell them? How many of these will he be able to sell?*
(iii) *Based on his experience in the second month he decides to produce 130,000 appliances in the third month. At what price will he sell them? How many of these will he be able to sell?*
(iv) *After several months of experimentation, he guesses that consumer demand is described by the linear (demand) equation given above. If he wishes to make just enough appliances so that they will all sell while meeting the full demand, how many will he produce per month? At what price will he sell them?*

EXERCISE 4.2. *Suppose that the supply equation of another appliance is given by* $p = .006q + 103.5$ *and assume that the manufacturer collects the following data: In the first month, he produces 5000 appliances and sells them all at $133.50 and believes that he could have sold over 4000 more. In the second month, produces 7000 and again sells them all, this time at $145.50; also this time he has an additional 786 unfilled orders. In the third month, he produces 8000 and sells them at $151.50. But, this month he has 1071 left over. Is there a linear demand equation consistent with the above data? If so, find it and compute the number of appliances the manufacturer should produce a month. If there is no linear demand equation, explain why there is none.*

As the above exercises indicate, the theory behind this supply and demand model is that, in the open market place, the price and production level will stabilize at the point of intersection of the supply and demand curves: for our original model, at the solution to the system:

$$p = -0.0004q + 160$$
$$p = 0.0005q + 50$$

One easily checks that the solution to the system gives a production level of 122,222 appliances and a price of $111.11 per appliance. A second justification of this theoretical expectation is based on the assumption that the price will be set by the market place and not the manufacturer. Again suppose that the time frame for this model is one month and that in the first month 100,000 appliances are produced and sold. Because of the high demand, the selling price rises to \$120 ($p = -0.0004(100,000) + 160 = 120$). Responding to this price, the manufacturer produces 140,000 appliances in the second month ($q = \frac{1}{0.0005}(120 - 50)$). But now the price drops to \$104.00 ($p = -0.0004(140,000) + 160$). In response to the drop in price, the third month's production level is set at 108,000 ($q = \frac{1}{0.0005}(104 - 50)$) resulting in a price increase to \$116.80, etc. This iterative process is illustrated in the following diagram:

q:	100,000		140,000		108,000		133,600		113,120		129,504		116,397		126,883	…
	↘	↗	↘	↗	↘	↗	↘	↗	↘	↗	↘	↗	↘	↗	↘	
p:		\$120.00		\$104.00		\$116.80		\$106.56		\$114.75		\$108.20		\$113.44		⋯

This process will eventually stabilize at $q = 122,222$ and $p = 111.11$. It is not surprising that this iterative process may be described by a linear difference equation. Let q_n denote the level of production in the nth month. Then the price in the $n - 1$st month would stabilize at $p_{n-1} = -0.0004q_{n-1} + 160$ and, as a result of this new price, the production level set for the next month, q_n, would be set at $\frac{1}{0.0005}((-0.0004q_{n-1} + 160) - 50)$. Simplifying, we get:

$$q_n = -\frac{4}{5}q_{n-1} + 220,000$$

which has fixed point:

$$c = \frac{220,000}{1 + \frac{4}{5}} = 122,222.222\cdots$$

EXERCISE 4.3. *Making up interesting examples that have reasonable solutions is always a problem for the teacher and textbook writer. One useful technique is to solve the problem symbolically and then choose the constants to give a nice example. For example, linear demand and supply equations always have the form:*

$p = -dq + e$ and $p = sq + t$, where d, e, s, t are all positive constants.

(i) *Solve the above system for p and q in terms of the constants d, e, s, and t.*
(ii) *Create a supply and demand model for a business firm which has an equilibrium values at about $q = 35,000$ and $p = 7.25$. Also, construct your models so that the supply and demand equations make sense in the production range $5,000 \leq q \leq 60,000$. Finally, try to construct your model so that the prices*

for both equations do not fluctuate more than \$2.00 in the given range.

4.2. Supply and Demand with Taxes

Supply and demand equations can be used to assess the effect of taxes on the production level and sales price of the product being modeled. There are basically two types of taxes: a *flat tax* of a fixed amount or a *percentage tax*, a fixed percentage of the sale price. Also the tax may be levied on the consumer as a *sales tax* or on the manufacturer as a *manufacturer's tax*. This gives four different options. However, in each case, the questions that must be considered by the government in deciding to impose a tax are the same:

(i) How much revenue will the tax yield for the government body levying the tax?
(ii) How much will the price of the product change?
(iii) How much will the production level change?

It is clear that the answer to (ii) will have a direct effect on the answer to (i). The importance of the answer to (iii) stems from its secondary effects: a significantly lower production level could result in lost jobs and/or wages for the workers and this, in turn, would result in less revenue since laid off workers pay less in taxes and may need public assistance.

Let us consider how taxes would work with the production and sale of the small appliance modeled above. So that we can compare all four options, let us consider a 5% tax or a flat tax of \$5.56 ($.05 \times 111.11$). The important thing to keep in mind as we consider the various options is that the variable p must always represent *the actual amount paid by the consumer to the manufacturer and q represents the number of items manufactured and sold*. We start with the flat sales tax.

Consider our example with a flat \$5.56 sales tax, i.e. \$5.56 to be paid by the consumer on top of the sale price p. Since the manufacturer is not involved in the taxation (except that he may collect it for the government), the supply equation remains the same. On the other hand, the number of appliances that will be bought will depend on the total amount paid by the consumer not simply p, the amount paid to the manufacturer. We must, therefore, alter the demand equation. The price p is replaced, in the demand equation, by $p + 5.56$, the amount that the consumer actually pays. Solving for p, then yields a new demand equation:

$$(p + 5.56) = -0.0004q + 160 \quad \text{or} \quad p = -0.0004q + 154.44.$$

Solving simultaneously the supply equation and this new demand equation gives a production level of 116,044 appliances at a price of \$108.02 each. The actual price paid by the consumer will be \$113.58 (108.02+5.56). The government will collect \$645,204.64 in taxes per month ($5.56 \times 116,044$). However, the level of production will drop by about 5% or 6,178 appliances per month to be exact.

Now suppose that the flat \$5.56 tax is imposed as a manufacturer's tax, i.e. \$5.56 to be paid by the manufacturer for each appliance sold. Since this time the consumer is not involved in the taxation, the demand equation remains the same and it is the supply equation which must be changed. The number of appliances that will be produced depends on the portion of amount paid by the consumer which will be retained by the manufacturer. The new supply equation is obtained by substituting $p - 5.56$ for p in the supply equation and solving for p:

$$(p - 5.56) = 0.0005q + 50 \quad \text{or} \quad p = 0.0005q + 55.56.$$

Solving simultaneously the new supply equation and the original demand equation gives a production level of 116,044 appliances at a price of \$113.58 each. Again, the government will collect \$645,204.64 in taxes per month and the level of production will drop by 6,178 appliances per month. It seems that whether a flat tax is imposed as a sales tax or a manufacturer's tax is immaterial, except, possibly, for political considerations. We will see shortly that this is not the case for a percentage tax.

Exercise 4.4. *On the same set of axes, plot the original and the new supply and demand equations. Explain geometrically why a flat tax has the same effect on production levels whether it is imposed on the consumer or the manufacturer.*

Now we consider a 5% sales tax, i.e. a tax of $.05p$ to be paid by the consumer in addition to the sale price p. Again the manufacturer is not involved in the taxation and the supply equation remains the same. As noted above, the number of appliances that will be sold will depend on the total amount paid by the consumer and we must alter the demand equation by substituting $p + (.05)p$ for p:

$$(p + .05p) = (1.05)p = -0.0004q + 160 \quad \text{or} \quad p = \frac{-0.0004}{1.05}q + \frac{160}{1.05}.$$

Solving simultaneously the supply equation and this new demand equation gives a production level of 116,216 appliances at a price of \$108.11 each. The actual price paid by the consumer will be \$113.52 (1.05×108.11). The government can expect to collect only \$628,728.56 in taxes per month, about \$15,500 less than with a flat tax. However, the level of production will drop by only 6,006 appliances per month, about 175 fewer than with the flat tax.

Finally, we consider a 5% manufacturer's tax. As before, the demand equation remains the same and it is the supply equation which must be changed. The new supply equation is:

$$(p - (.05)p) = (.95)p = 0.0005q + 50 \quad \text{or} \quad p = \frac{0.0005}{.95}q + \frac{50}{.95}.$$

Solving simultaneously the new supply equation and the original demand equation gives a production level of 115,909 appliances at a price of \$113.64 each. this time, the government collects \$658,581.94 in taxes per month and the level of production will drop by 6,313 appliances per month. (In computing the total tax collected for the sales tax, $.05 \times 108.11$ is rounded first and then multiplied by 116,216; no rounding takes place when the total manufacturer's tax is computed, since the manufacturer is not paying the tax appliance by appliance.)

In the case of a percentage tax, the price to the consumer is much the same whether the tax is a sales tax or a manufacturer's tax. From the government point of view, the manufacturer's tax produces significantly more in revenue. But, on the down side, the manufacturer's tax results in a significantly lower production level. How does the manufacturer, as opposed to his employees, fare under these tax schemes? To answer this question, we must consider revenue, production costs and profits - the subjects of our next investigation.

Exercise 4.5. *On the same set of axes, plot the original and the new supply and demand equations. Explain geometrically why a percentage tax behaves differently when imposed on the consumer and the manufacturer.*

Exercise 4.6. *Rework these examples with a 9% tax or a flat \$10.00 tax.*

Exercise 4.7. *Consider the general model discussed in Exercise* **??**.

(i) *Compute the new equilibrium point when a flat tax of f is imposed on the consumer.*
(ii) *Compute the new equilibrium point when a percentage tax of f is imposed on the consumer.*

4.3. Revenue, Production Costs and Profits

Considering our small appliance manufacturer, we may ask about the profits that the firm will make under various conditions. Profit is simply revenue (the amount taken in) minus costs. So our first step in discussing profits is to consider revenue. Revenue, in turn, is simply selling price times quantity sold: $p \times q$, as we have defined those variables. In our original example, the manufacturer produced and sold 122,222 appliances per month at \$111.11 each. The manufacturer's revenue is therefore \$13,580,086.42 per month. With a \$5.56 sales tax his revenue is about one million less, $\$108.02 \times 116,044 = \$12,535,072.88$, to be exact. If this flat tax is imposed as a manufacturer's tax, we compute $\$113.58 \times 116,044 = \$13,180,277.52$. But, subtracting the \$645,204.64 in taxed passed directly to the government, we again get \$12,535,072.88, which should come as no surprise. The one million in lost revenue due to the tax is offset by a smaller production level and, one assumes a lower production cost. So, at this point, it is still not clear what the effect of the tax will be on profits.

EXERCISE 4.8. *Compute the revenues for our small appliance manufacturer when a 5% sales tax or manufacturer's tax is imposed.*

It is very natural to expect a linear model for production costs. Production costs often fall into two categories: *fixed costs* and *variable* or *marginal costs.* Examples of fixed costs include rent on the factory, equipment maintenance costs and many management costs; examples of variable costs include raw materials, equipment operating costs and most labor costs. Fixed costs are reported as a single figure while marginal costs are reported as a per item amount. Suppose that it costs \$755,000 a month in fixed costs to keep our small appliance factory running and that raw materials, labor costs and other variable costs come to \$101.50 per appliance. The resulting cost function is then: $c = c(q) = (101.5)q + 755,000$. Now that we have the cost function we may now compute the company's profits under the various tax schemes discussed above. Before a tax was imposed, the monthly revenue was just computed to be \$13,580,086.42. The monthly costs are $c(122,222) = 101.5 \times 122,222 + 755,000 = 13,160,533$. Thus, if no tax is imposed, the company will show a profit \$419,553.42 each month. If the flat tax (sales or manufacturer's) is imposed, revenue was computed above to be \$12,535,072.88 per month. The monthly production cost at the new lower production level is $c(116,044) = 12,533,466$. The result is a dramatic drop in profit to only \$1,616.88 per month!

EXERCISE 4.9.

(i) *Compute the monthly production cost and profit for this company when a 5% tax is imposed. Which is better for the manufacturer a 5% sales tax or a 5% manufacturer's tax?*

(ii) *Without computing, what do you think the effect of a 9% or \$10.00 tax would be no matter how it is imposed? Check out your guess.*

Exercise 4.10. *Consider our small appliance factory and assume that a 5% sales tax is in effect. Suppose the manufacturer has \$1,000,000 to invest in his company. He has three options: the \$1,000,000 could be used to relocate to a newer and smaller factory with lower rent and easier maintenance; it could be used to buy new computer-guided equipment on the production line; or it could be used to hire a lobbying firm to try to convince the government to reduce the present 5% manufacturer's tax. He believes that the move to the new location would result in reducing his monthly fixed costs by \$250,000, that the new equipment on the production line would reduce his marginal costs by \$2.10, and that the lobbying firm could get the tax reduced from 5% to 2.5%. Which option should he choose?*

4.4. Marginal Revenue and Marginal Profit

As we have described it, the demand equation is a "fact of life" for the manufacturer, that is, it is not something over which he has any control. Without making changes in his operation, the same can be said of his cost function. The supply function, on the other hand, is a function of his own invention. In this section, we wish to consider some of the analysis carried out in developing a supply equation. One important feature of any business operation is profit and maximizing profit is certainly a major concern of the manufacturer.

Maximizing a function is usually considered a problem in the domain of calculus. But there is a tool in in the mathematics of business which can often be used in place of calculus: marginal functions. Suppose that we have a profit function $P(q)$ where, as above, q represents the quantity of items. Suppose that 1000 items are sold. Then the marginal profit at the production level 1000 is the *additional* profit that would be made if one more item were sold: $f(1001) - f(1000)$. Of course, we can compute the marginal profit at any production level: $f(q + 1) - f(q)$. This results in a new function of the variable q, $f^*(q) = f(q+1) - f(q)$, called the marginal profit function. Of course, in this computation, the fact that f is a profit function has no significance and, using this definition, we may consider the marginal function f^* of any function f.

In our ongoing manufacturing example, the cost function is a linear function and therefore the marginal cost is simply the slope of this function. This is true of any linear function $f(q) = aq + b$:

$$f^*(q) = f(q+1)-f(q) = (a(q+1)+b)-(aq+b) = aq+a+b-aq-b = a.$$

The marginal cost should be interpreted as the additional cost of producing *one more* appliance. Note that the marginal cost is *not* the cost of producing *one* appliance. The cost of producing one appliance is called the *average* cost; it is denoted by $\bar{c}$; it is given by $\bar{c} = \bar{c}(q) = \frac{c(q)}{q}$ and it depends on the number of appliances produced. The average cost of producing one item is an important bit of information for the over all management of the company. But, for the specific decision as to whether the present production level should be increased, maintained or decreased, it is the marginal cost that is most important.

EXERCISE 4.11. *Let c be the linear cost function $c(q) = aq + b$.*

(i) *Compute the average cost function $\bar{c}(q) = \frac{c(q)}{q}$. How are the marginal and average costs related for a linear cost function?*
(ii) *In the case of our small appliance factory, graph the marginal and average cost functions on the same axes.*

When $f(q)$ is a linear function of q, then $f^*(q)$ is a constant equal to the slope of $f(q)$. When $f(q)$ is not a linear function, then $f^*(q)$ is not a constant function. Nevertheless, $f^*(q)$ can be interpreted as a slope. It is the slope of the line through the points $(q, f(q))$ and $(q + 1, f(q + 1))$: $\left(f^*(q) = \frac{f(q+1)-f(q)}{(q+1)-q}\right)$. In many cases, $f^*(q)$ is very close to the slope of the tangent line to the graph of $f(q)$ at the point $(q, f(q))$. In fact, if the graph of the function $f(q)$ is "smooth", then $f^*(q)$ is the slope of the tangent line to the graph of $f(q)$ at some point between $(q, f(q))$ and $(q+1, f(q+1))$. This follows from an important theorem from calculus called "The Mean Value Theorem". If you draw a few examples for yourself you will see that this is not a surprising result.

EXERCISE 4.12. *Let $f(q) = \frac{2}{5}q^2 - q - 1$*

(i) *Compute $f^*(q)$.*
(ii) *Graph $f(q)$, graph the tangent to this curve at $(q, f(q))$ and graph the line through $(q, f(q))$ and $(q + 1, f(q + 1))$ when $q = 2$.*
(iii) *Repeat* **b** *for $q = 3$ and $q = 4$.*

Let us compute the marginal revenue and marginal profit for our small appliance manufacturer. As we have already noted, revenue is simply price times quantity sold. If q items are put on the market in a given month, the price at which they will all sell is determined by the demand equation. Hence, the revenue function is:

$$r = r(q) = pq = (-0.0004q + 160)q = -0.0004q^2 + 160q.$$

In the following table we compute the revenue at several production levels:

quantity	50,000	100,000	150,000	200,000	250,000
revenue	\$7,000,000	\$12,000,000	\$15,000,000	\$16,000,000	\$15,000,000

Revenue seems to peak at about the production level of 200,000. We can verify this by considering the marginal revenue:

$$\begin{aligned} r^*(q) &= r(q+1) - r(q) \\ &= \big(-0.0004(q+1)^2 + 160(q+1)\big) - \big((-0.0004q^2 + 160q\big) \\ &= -0.0008q + 159.9996. \end{aligned}$$

Rounding this to the nearest cent, gives $r^*(q) = -0.0008q + 160$. At $q = 100,000$, we have a marginal revenue of \$80. This means that by producing one more appliance, 100,001 in all, the revenue will increase by \$80. At $q = 150,000$ the marginal revenue has dropped to \$40, at $q = 200,000$ it is zero and at $q = 250,000$ it is -\$40. In this last case, the production of one more appliance will result in a lost of revenue. In general, if $q < 200,000$ and production is increased, revenue will increase while if $q > 200,000$ and production is increased, revenue will decrease. Thus, revenue can be maximized by setting the production level at 200,000.

Now let's turn our attention to the profit function. The profit function is defined to be revenue minus cost; so, the profit function is:

$$\begin{aligned} P = P(q) &= r(q) - c(q) = (-0.0004q^2 + 160q) - (101.50q + 755,000) \\ &= -0.0004q^2 + 58.50q - 755,000. \end{aligned}$$

(We use capital P for profit to distinguish if from the price p.)

We now append profit to the above table:

quantity	50,000	100,000	150,000	200,000	250,000
revenue	\$7,000,000	\$12,000,000	\$15,000,000	\$16,000,000	\$15,000,000
profit	\$1,170,000	\$1,095,000	-\$980,000	-\$5,055,000	-\$11,130,000

When revenue is at its highest (\$16,000,000 at $q = 200,000$), the company will have a loss of over \$5,000,000. The reason for this is clear: at this production level the sale price of each appliance is only \$80 while it cost slightly more than \$101.50 to produce that appliance! Profit seems to peak when production is low. To find out exactly where profit is maximum we compute the marginal profit. The easy way to do this is to observe that since profit equals revenue minus cost, marginal profit equals marginal revenue minus marginal cost. We'll use this formula now and you will justify it later:

$$P^*(q) = r^*(q) - c^*(q) = (-0.0008q + 160) - (101.50) = -0.0008q + 58.50.$$

One easily checks that the cutoff production level is 73,125: if fewer than 73,125 appliances are produced, marginal profit is positive and producing another appliance will increase profit; while, if more than 73,125 appliances are produced, marginal profit is negative and producing one more appliance will decrease profit. In this last case, producing one less appliance will usually increase profit. We conclude that the maximum profit possible is \$1,383,906.25 and that will occur when 73,125 appliances are produced.

At this point it is quite natural to ask why this analysis yields a different production level and price from the one obtained using the supply and demand model. Since we are using the same demand equation, the explanation of this discrepancy must involve the supply equation. The fact is that the supply equation is the result of several different considerations, immediate profits being only one of them. There may be several reasons why a company may wish increase production levels beyond that which maximizes profits. For example, maintaining its market: At a production level of 73,125 appliances the price is \$130.75. At this high a price another manufacturer may be tempted to start producing this appliance. Suppose that a competitor does enter the market and produces 49,097 appliances a month. Since 122,222 appliances are produced the price drops to \$111.11. Our manufacturers revenue drops to \$8,124,918.75; while his production costs drop to \$8,177,187.50. The result would be a monthly loss of \$52,268.75. Other considerations that may go into the supply equation include contractual agreements with employees and the size of the physical plant.

EXERCISE 4.13. *Consider the demand equation discussed in Exercise* **?? a** *and consider the general linear cost function* $c(q) = mq + f$, *where* m *is the marginal cost and* f *the fixed cost.*

(i) *Compute the revenue function.*
(ii) *Compute the profit function.*
(iii) *Compute the marginal profit function.*
(iv) *At what production levels will the profit be maximized?*

(v) *Select values for m and f which are consistent with the example you produced in Exercise* **?? b**. *Arrange it so that your company makes a profit and so that the maximum profit occurs in the production range of the model.*

We close this section with exercises on the calculation and interpretation of marginal functions. The reader familiar with Calculus should be struck by the similarity between this "calculus" (method of calculating) for marginal functions and the "calculus" for derivatives.

EXERCISE 4.14. *Let $g(q)$ and $h(q)$ be two functions of the quantity q.*

(i) *Show that, if $f(q) = g(q) \pm h(q)$, then $f^*(q) = g^*(q) \pm h^*(q)$.*
(ii) *Show that, if $f = g \times h$, then*

$$f^*(q) = h(q+1) \times g^*(q) + h^*(q) \times g(q) \quad \text{and}$$
$$f^*(q) = h(q) \times g^*(q) + h^*(q) \times g(q+1).$$

(iii) *Show that, if $f = \frac{g}{h}$, then*

$$f^*(q) = \frac{g^*(q) \times h(q) + g(q) \times h^*(q)}{h(q+1) \times h(q)}.$$

The reader familiar with Calculus should note that the "chain rule" is missing.

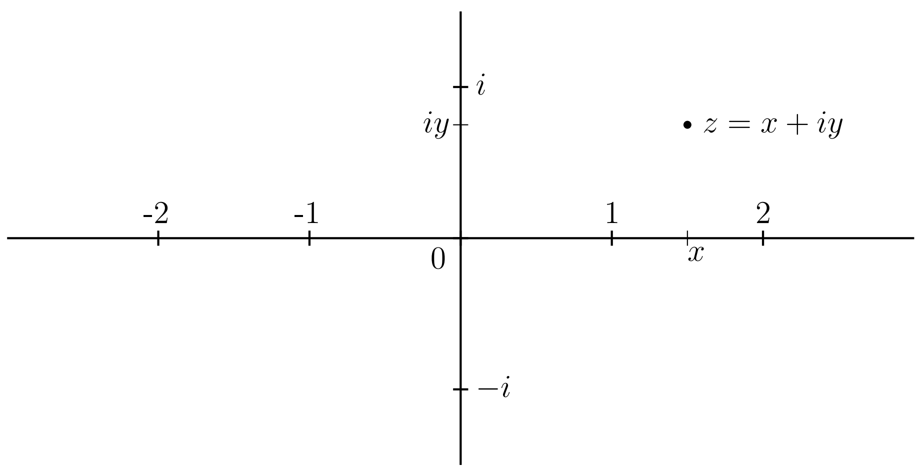
EXERCISE 4.15. *Explain just why the "chain rule" is missing.*

CHAPTER 5

The Complex linear Function

5.1. The Complex Numbers

We will use the standard notation for the complex numbers: $z = x + iy$ where x, y are real numbers and $i^2 = -1$, i.e., i denotes a square root of -1. The real number x is called the "real" part of z while iy is called the "imaginary" part of z. If $y = 0$, z is a real number and so the real numbers form a subset of the complex numbers. If $x = 0$, $z = iy$ and is called a "pure" imaginary number. As is usual, we will visualize the complex numbers as the points of the "complex plane":

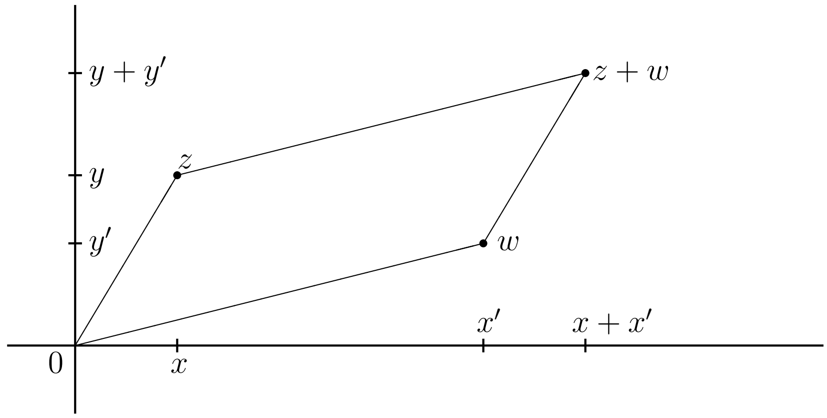


In this picture, the real numbers form the horizontal line, called the *real axis,* and the pure imaginary numbers form the vertical line, called the *imaginary axis.*

If $z = x + iy$ and $w = x' + iy'$ are two complex numbers, then $z + w$ is computed as follows:

$$z + w = x + iy + x' + iy' = (x + x') + i(y + y').$$

In the next figure, we recall the geometry of complex addition: the sum $(z + w)$ completes a parallelogram with 0, z and w and lies opposite 0 on this parallelogram.



Fix the complex number w and consider the transformation of the complex plane which maps each complex number z onto $z + w$. Referring again to the above figure, we see that this transformation is translation, in fact, the translation which maps 0 to w.

Exercise 5.1.

(i) *Give a geometric description the translation given by adding* 1 *to each complex number.*
(ii) *Describe the translation given by adding* i *to each complex number.*
(iii) *Describe the translation given by adding* $1 - i$ *to each complex number.*

In the first four chapters, we assumed a facility with the algebra of the real numbers. We are now going to want to carry out algebraic manipulations with complex numbers and so we have to make sure that the complex numbers satisfy the basic "rules of algebra." But first, we review these rules.

Consider the set of real numbers, $\mathbb{R}$, along with the two operations addition, $+$, and multiplication, $\times$. For all $r, s, t \in \mathbb{R}$:

(i) (Associativity of $+$) $(r + s) + t = r + (s + t)$;
(ii) (Commutativity of $+$) $r + s = s + r$;
(iii) (Additive identity) There exists a real number 0 so that $r + 0 = 0 + r = r$;
(iv) (Additive inverse) There exists a real number $-r$ so that $r + (-r) = (-r) + r = 0$;
(v) (Associativity of $\times$) $(r \times s) \times t = r \times (s \times t)$;
(vi) (Commutativity of $\times$) $r \times s = s \times r$;
(vii) (Multiplicative identity) There exists a real number 1 so that $r \times 1 = 1 \times r = r$;

(viii) (Multiplicative inverse) If $r \neq 0$, there exists a real number $\frac{1}{r}$ so that $r \times \frac{1}{r} = \frac{1}{r} \times r = 1$;

(ix) (Distributivity) $r(s+t) = rs + rt$ and $(r+s)t = rt + st$.

These "rules of algebra" are called the *field axioms* and any set with two operations denoted by $+$ and $\times$ that satisfy these axioms is called a field. The set of real numbers $(\mathbb{R}, +, \times)$ is a field. Since adding or multiplying two rational numbers yields another rational number and since, for any rational number $r \neq 0$, $-r$ and $\frac{1}{r}$ are also rational, the set of rational numbers $(\mathbb{Q}, +, \times)$ also satisfy the field axioms. On the other hand, the integers $(\mathbb{Z}, +, \times)$ do not form a field. $\mathbb{Z}, +, \times$ satisfies all of the field axioms except (viii): With the exception of 1 and -1, no number in $\mathbb{Z}$ has a multiplicative inverse (in $\mathbb{Z}$). As you will soon show, the set of complex numbers $(\mathbb{C}, +, \times)$ is a field; the field axioms for $\mathbb{C}$ follow easily from the field axioms for $\mathbb{R}$. We defined the addition of complex numbers above. For complex numbers $z = x + iy$ and $w = x' + iy'$, we define the product zw follows:

$$zw = (x+iy)(x'+iy') = xx'+ixy'+iyx'+i^2yy' = (xx'-yy')+i(xy'+yx').$$

LEMMA 4. *For all $z, v, w \in \mathbb{C}$:*

(i) *(Associativity of +) $(z+v)+w = z+(v+w)$;*

(ii) *(Commutativity of +) $z + w = w + z$;*

(iii) *(Additive identity) There exists a complex number 0 so that $z + 0 = 0 + z = z$;*

(iv) *(Additive inverse, There exists a complex number $-z$ so that $z + (-z) = (-z) + z = 0$;*

(v) *(Associativity of $\times$) $(z \times v) \times w = z \times (v \times w)$;*

(vi) *(Commutativity of $\times$) $z \times w = w \times z$;*

(vii) *(Multiplicative identity) There exists a complex number* 1 *so thatz $\times 1 = 1 \times z = z$;*

(viii) *(Multiplicative inverse) If $z \neq 0$, there exists a complex number $\frac{1}{z}$ so that $z \times \frac{1}{z} = \frac{1}{z} \times z = 1$;*

(ix) *(Distributivity) $z(v+w) = zv + zw$ and $(z+v)w = zw + vw$.*

PROOF. Most parts of this proof are routine and are left as an exercise. Here we simply verify (viii). We note that he zero complex number is simply the real number 0 interpreted as a complex number: $0 = 0 + i0$. Similarly the multiplicative identity is the real number 1: $1 = 1 + i0$. For the multiplicative inverse of $z = x + iy$, we need a complex number $x' + iy'$ so that $(x+iy)(x'+iy') = 1$. The complex number that works is $\frac{(x-iy)}{x^2+y^2}$ (note that since $z \neq 0$, $x^2 + y^2 \neq 0$). We have:

$$(x+iy)\frac{(x-iy)}{x^2+y^2} = \frac{(x+iy)(x-iy)}{x^2+y^2} = \frac{x^2+y^2}{x^2+y^2} = 1$$

□

EXERCISE 5.2. *Complete the proof of Lemma* **??**.

Now let's look at the geometry of the multiplication of complex numbers. Consider zw, where z is the real number 3. If $w = x+iy$, $3w = 3x+i(3y)$ and we see that $3w$ lies on the line through 0 and w and is three times the distance from zero as is w. In geometric terms, we see that multiplication by the (real) complex number 3 is a "dilation" of the complex plane with center 0 and "magnification" 3: each complex number is moved directly away from the origin so that its distance from 0 is multiplied by 3.

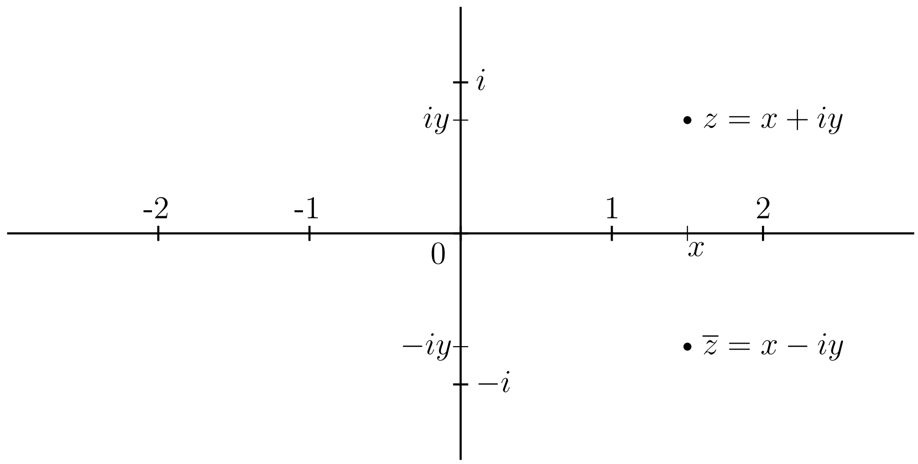
Next, consider multiplying each complex number by -1. It is tempting to describe this geometrically as reflecting each complex number through 0, the origin. However, as we will explain later, this is temptation should be resisted. Instead, we should think of multiplying each complex number by -1 as rotation the complex plane by 180 degrees about 0. We would also call this transformation the *half-turn* about 0. Thinking of multiplying by -1 as a rotation is consistent with multiplication by i: geometrically $i \times (x+iy) = -y+ix$ is the result of rotating $(x+iy)$ counterclockwise by 90 degrees about 0.

EXERCISE 5.3.

(i) *Describe geometrically (in terms of a transformation of the complex plane) the effect of multiplying each complex number by* $\frac{1}{3}$.
(ii) *Describe geometrically the effect of multiplying each complex number by any fixed positive real number.*
(iii) *Verify for each of the following complex numbers that multiplication by i rotates them counterclockwise by 90 degrees about 0: 1, 2, i, -7, 1+i, 1+2i.*
(iv) *Describe geometrically the effect of multiplying each complex number by* -2.
(v) *Describe geometrically the effect of multiplying each complex number by* $-i$.

In view of these examples, it seem that multiplying by an arbitrary complex number should correspond to some combination of a dilation and a rotation about 0. The precise geometric interpretation of multiplication by a arbitrary complex number is best explained in terms of the trigonometric representation of that complex number. So we leave the geometric interpretation of multiplication for now and turn to complex conjugation.

The complex conjugate of the complex number $z = x + iy$ is defined to be $x - iy$ and is denoted by $\overline{z}$. Geometrically, $\overline{z}$ is the reflection of z through the real axis:



In the next lemma, we list the fundamental properties of the complex conjugate. Each of these properties follows from the definitions and simple algebraic manipulations.

LEMMA 5. *For any complex numbers z and w,*

(i) $\overline{\overline{z}} = z$;
(ii) $\frac{z+\overline{z}}{2}$ *and* $\frac{z-\overline{z}}{2i}$ *are both real and* $z = \left(\frac{z+\overline{z}}{2}\right) + i\left(\frac{z-\overline{z}}{2i}\right)$;
(iii) $\overline{zw} = \overline{z}\ \overline{w}$;
(iv) $\overline{z + w} = \overline{z} + \overline{w}$.

EXERCISE 5.4.

(i) *Give a geometric explanation for items (i) and (iv) in the previous exercise.*
(ii) *Draw a picture similar to the one illustrating the definition of the complex conjugate. In addition to z and $\overline{z}$, plot $-\overline{z}$, $z+\overline{z}$, $z - \overline{z}$, $\frac{z+\overline{z}}{2}$ and $\frac{z-\overline{z}}{2}$. From this picture verify item (ii) in the previous exercise.*

EXERCISE 5.5. *Fill in the "algebraic" proofs for Lemma* **??**.

EXERCISE 5.6. *Describe each of the following transformations geometrically:*

(i) $z \to -\overline{z}$;
(ii) $z \to i\overline{z}$;
(iii) $z \to -i\overline{z}$.

Another simple but important fact about the complex conjugate is that the product $z\overline{z}$ is a nonnegative real number: if $z = x + iy$, then $z\overline{z} = (x+iy)(x-iy) = x^2 - i^2y^2 = x^2+y^2$. From this, we see that $z\overline{z}$ is actually positive whenever z is different form zero. Continuing this line of thought, we see that $\frac{1}{z}$ is simply the complex number $\overline{z}$ multiplied by the positive real number $\frac{1}{z\overline{z}}$:

$$\left(\frac{\overline{z}}{z\overline{z}}\right) z = z\left(\frac{\overline{z}}{z\overline{z}}\right) = \frac{z\overline{z}}{z\overline{z}} = 1.$$

It follows that the quotient of two complex numbers is well defined: if w and $z \neq 0$ are complex numbers, we have $\frac{w}{z} = \frac{w\overline{z}}{z\overline{z}}$, which is the complex number $w\overline{z}$ multiplied by the positive real number $\frac{1}{z\overline{z}}$.

EXERCISE 5.7. *Describe geometrically the set of complex numbers z such that $\frac{1}{z} = \overline{z}$.*

If lengths are measured in the usual way along the real and imaginary axis, we can compute the distance from 0 to z using the Theorem of Pythagoras. One easily checks that the distance from 0 to $z = x+iy$ equals $\sqrt{x^2+y^2}$ which in turn equals $\sqrt{z\overline{z}}$. Hence we define the *modulus* or *length* of $z = x + iy$ by: $|z| = \sqrt{z\overline{z}}$. In the next lemma we list the fundamental properties of the modulus.

LEMMA 6. *For any complex numbers z and w,*

(i) $|\overline{z}| = |z|$;
(ii) $|-z| = |z|$;
(iii) $|zw| = |z||w|$;
(iv) $|z+w| \leq |z| + |w|$, *(the triangle inequality).*

PROOF.

(i) $|\overline{z}| = \sqrt{\overline{z}\,\overline{\overline{z}}} = \sqrt{\overline{z}z} = |z|$.
(ii) $|-z| = \sqrt{-z(\overline{-z})} = \sqrt{z\overline{z}} = |z|$.
(iii) $|zw| = \sqrt{zw\overline{zw}} = \sqrt{z\overline{z}}\sqrt{w\overline{w}} = |z||w|$.

(iv) Proving the triangle inequality is a bit tricky. We start by expanding $|z+w|^2$:

$$|z+w|^2 = (z+w)(\overline{z+w}) = z\overline{z}+z\overline{w}+w\overline{z}+w\overline{w} = |z|^2+|w|^2+(z\overline{w}+w\overline{z}).$$

Now, the trick is to recognize $z\overline{w} + w\overline{z}$ as twice the real part of $z\overline{w}$ (Lemma **??**(ii)):

For any complex number $a = x + iy$,

$$a + \overline{a} = 2x \leq 2\sqrt{x^2+y^2} = 2|a|.$$

Thus:

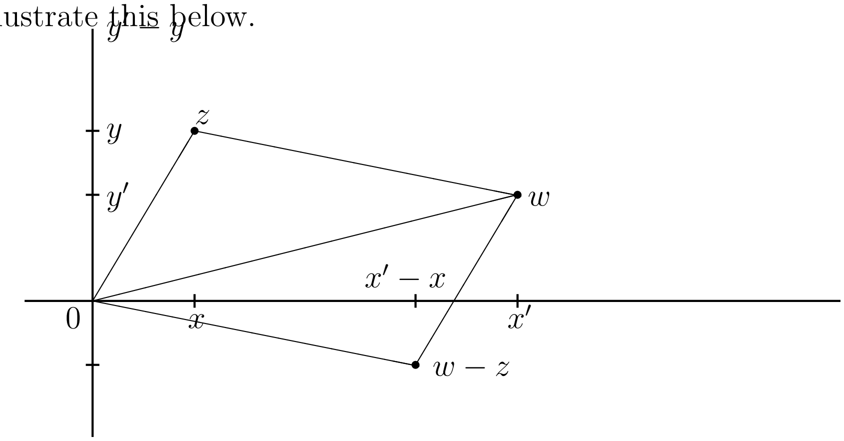
$$|z+w|^2 = |z|^2 + |w|^2 + (z\overline{w} + w\overline{z}) \leq |z|^2 + |w|^2 + 2|z\overline{w}|,$$

and

$$|z + w|^2 \leq |z|^2 + |w|^2 + 2|z||w|.$$

Finally, taking the positive square root of both sides gives the triangle inequality. □

The general concept of length can also be defined in terms of the modulus. The distance between z and w is defined to be $|w - z|$. We illustrate this below.



Now we may explain the reason Lemma **??**d is called the triangle inequality. Let z, z' and z'' be any three complex numbers. Let $a = z - z'$ and $b = z' - z''$, then $a + b = z - z''$ and:

$$|z - z''| = |a + b| \leq |a| + |b| = |z - z'| + |z' - z''|.$$

In this form, we see that the triangle inequality may be interpreted to mean that, in the complex plane, the length of any one side of a triangle is less than of equal to the sum of the lengths of the other two sides.

The triangle inequality leads directly to the definition of the segment joining v and w. The point z is on the segment joining v and w if $|v - z| + |w - z| = |v - w|$. This in turn enables us to define v, w and z to be colinear whenever one of these points is on the segment joining the other two. And finally, we define the line through v and w to be the set of all points z so that v, w and z are colinear. With these definitions, we may take the complex plane as a model of the euclidean plane, i.e., we may verify Euclid's axioms. This definition of a line is rather difficult to use; the parametric equation for the line through v and w which you will work out in the next exercise is very useful.

EXERCISE 5.8. *Let $v = a + ib$, $w = c + id$ and $z(t) = (ta + (1 - t)c) + i(tb + (1 - t)d)$ for the real variable t.*

(i) *Show that $z(0) = w$ and $z(1) = v$.*
(ii) *Compute $|z(t) - v|$ and $|z(t) - w|$ as multiples of $|v - w|$.*

(iii) *Prove that* v, w *and* $z(t)$.
(iv) *Describe the location of* $z(t)$ *relative to* v *and* w *in terms of* t.
(v) *Show that every point on the line through (as defined above) equals* $z(t)$ *for some* t.

Continuing with this geometric point of view, we note that circles are easy to describe: the set of complex numbers z satisfying $|z-c| = r$, where c is a fixed complex number (the center) and r is a fixed positive real number (the radius). One circle is particularly important. In Exercise **??**, you should have concluded that the complex numbers which have their complex conjugate as their multiplicative inverse form the circle of radius 1 about 0. We verify this by observing that z has $\overline{z}$ as its multiplicative inverse if and only if $z\overline{z} = 1$ and that $z\overline{z} = 1$ if and only if $|z-0| = |z| = 1$. We will call the circle given by $|z| = 1$ the *unit circle* of the complex plane. For easy reference, we formally state:

LEMMA 7.

(i) *The unit circle consists of the complex numbers which have their complex conjugate as their multiplicative inverse.*
(ii) *For any complex number* z, $\frac{z}{|z|}$ *lies on the unit circle.*
(iii) *Each complex number* z *has a unique representation of the form* $z = ru$, *where* r *is a nonnegative real number and* u *is a complex number on the unit circle.*
(iv) *The unit circle is closed under multiplication: if* u *and* u' *lie on the unit circle, then so does* uu'.

EXERCISE 5.9. *Complete the proof of Lemma* **??**.

EXERCISE 5.10.

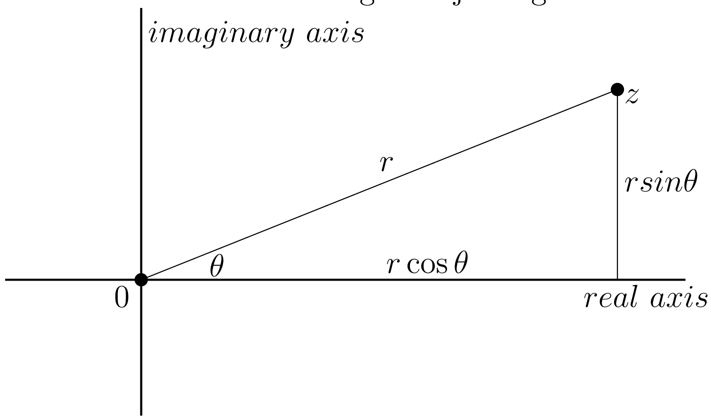
(i) *Describe the set of all complex numbers which are their own complex conjugate* ($z = \overline{z}$).
(ii) *Describe the set of all complex numbers which are their own multiplicative inverse* ($z = \frac{1}{z}$).

5.2. Complex Numbers and Trigonometry

The formula $z = x + iy$ may be viewed as representing the complex number in rectangular coordinates. In this coordinate system addition of complex numbers is simple component or vector addition. Alternatively, we may employ polar coordinates to specify a complex number. As we shall soon see, multiplication of complex numbers is particular easy to understand in the polar coordinate system.

The polar coordinates for z are the modulus of z and the angle θ that the line joining 0 and z makes with the positive real axis. Let $z = x + iy$, then $z\overline{z}$ is the square of the distance from 0 to z: $r^2 = z\overline{z} =$

$x^2 + y^2$. So, without loss of generality, we may write $x = r\cos\theta$ and $y = r\sin\theta$ for some angle θ between 0 and 2π radians. Then $r = |z|$ and θ is the (counterclockwise) measure of the angle (in radians) between the positive real axis to the line segment joining 0 to z.



One of the main advantages of the trigonometric form for complex numbers is that this notation makes the multiplication of complex numbers easier to understand:

THEOREM 7. *If* $z = r(\cos\theta + i\sin\theta)$ *and* $w = s(\cos\phi + i\sin\phi)$, *then* $zw = rs(\cos(\theta+\phi) + i\sin(\theta+\phi))$.

PROOF. By direct multiplication and the trigonometric sum formulas, we have:

$$\begin{aligned} zw &= rs(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= rs[(\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \cos\phi\sin\theta)] \\ &= rs[\cos(\theta+\phi) + i\sin(\theta+\phi)] \end{aligned}$$

□

This theorem enables us to interpret multiplication in terms of rotations and dilations. Let a be the complex number $r(\cos\theta + i\sin\theta)$ and let z be any other complex number. The complex number az may be constructed from z by first rotating z counterclockwise about the origin by an angle of measure θ and then multiplying its length by r. For example, the complex number i has polar form $i = \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})$; hence, as you noted in Exercise **??**, iz is obtained by rotating z counterclockwise $\frac{\pi}{2}$ radians about the origin.

EXERCISE 5.11. *Let* $z = r(\cos\theta + i\sin\theta)$ *and prove the following two formulas.*

(i) $\overline{z} = r(\cos(-\theta) + i\sin(-\theta))$
(ii) $z^{-1} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta))$

The polar form is also useful in showing that every complex number has a square root:

LEMMA 8. *Let* $z = r(\cos\theta + i\sin\theta)$ *and let* $w^2 = z$. *Then*

$$w = \pm\sqrt{r}(\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2})).$$

PROOF. Assume that $w = s(\cos\phi + i\sin\phi)$. Then, by Theorem **??**, $w^2 = s^2(\cos(2\phi) + i\sin(2\phi))$. Hence, if $w^2 = z$, $s^2 = r$ and $2\phi = \theta$ (mod 2π). Thus, $s = \pm\sqrt{r}$ and either $2\phi = \theta$ or $2\phi = (\theta + 2\pi)$. So either $w = \pm\sqrt{r}(\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2}))$ or $w = \pm\sqrt{r}(\cos(\frac{\theta}{2} + \pi) + i\sin(\frac{\theta}{2} + \pi)) = \mp\sqrt{r}(\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2}))$ □

For any complex number z, we will let $\sqrt{z}$ denote the solution to $w^2 = z$ which, in polar form, has the smallest nonnegative angle; $-\sqrt{z}$ is then the other solution. Note that if z is a positive real number, this agrees with the convention for real square roots.

EXERCISE 5.12. *Find the three cube roots of* $z = r(\cos\theta + i\sin\theta)$

Of particular interest are the "nth roots of unity", that the nth roots of the complex number 1. They are: 1, $(\cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n}))$, $(\cos(\frac{4\pi}{n}) + i\sin(\frac{4\pi}{n}))$, $\cdots$, $(\cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n}))$, $\cdots$, $(\cos(\frac{(2n-2)\pi}{n}) + i\sin(\frac{(2n-2)\pi}{n}))$.

EXERCISE 5.13.
(i) *Show that* $[\cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n})]^n = 1$, *for any integer* k.
(ii) *Show that if* $[\cos(\theta) = i\sin(\theta)]^n = 1$, *then* $\theta = \frac{2k\pi}{n}$, *for some integer* k.

EXERCISE 5.14. *Find all solutions to* $z^4 = -1$

EXERCISE 5.15. *Let* c *be an arbitrary complex number and show that, if* $u = (\frac{c}{\bar{c}})$, *then* u *lies on the unit circle and* $c = \pm|c|\sqrt{u}$.

5.3. The Complex Plane

In Chapter 2, we used the real number line as a model for the euclidean line to study the similarities of the euclidean line. In this chapter, we use the complex plane as a model of the euclidean plane to study the similarities of the euclidean plane. The key to this model is the formula for the distance between two points (complex numbers). Using this definition of distance, we restate the definitions given at the end of Chapter 1 in terms of the complex plane: A function f from the complex plane onto itself is a *similarity* if there is a positive real number m so that $|f(z) - f(w)| = m|z - w|$, for all complex numbers

z and w. The constant m is called the *magnification* of f. If $m = 1$, f is called a *congruence.*

The sequence of results in this chapter will be almost identical with the sequence of results in Chapter 2. In most cases, the proofs of the two dimensional results will be almost identical with the corresponding one dimensional proofs. To stress the similarity, we will indicate with each result the corresponding result from Chapter 2.

LEMMA 9. *[Lemma* **??***] Let $f(z) = az + b$ be a linear function. Then f is a similarity with magnification $|a|$.*

PROOF. We must show that f satisfies the definition of a similarity: Specifically, we must show that there is a positive real number m so that, for any two points z and w, the distance between $f(z)$ and $f(w)$ is m times the distance between z and w.

By direct computation:

$$\begin{aligned} |f(z) - f(z')| &= |(az - b) - (aw - b)|, \\ &= |az - aw|, \\ &= |a(z - w)|, \\ &= |a||z - w| \end{aligned}$$

The last step follows from Part (3) of Lemma **??**. We conclude that f is a similarity with magnification $|a|$. □

It is natural, at this point, to assume that we could prove the 2-dimensional analogue of Theorem **??**: all similarities are linear functions. But, this is not the case. For example, conjugation $z \to \overline{z}$, the reflection through the real axis, is a congruence that cannot be given by a linear function. However, a restricted analogue to Theorem **??** does hold. Before we can state this restricted result, we must introduce a new concept. Let z_0, z_1 and z_2 be three noncollinear points in the complex plane, i.e., the vertices of a triangle. If moving from z_0 to z_1 to z_2 and back to z we traverse the perimeter of that triangle clockwise (counterclockwise), we sat that the ordered triple (z, z_1, z_2) has a *clockwise (counterclockwise) orientation.* A similarity f is said to be *direct* if, for each ordered noncollinear triple (z, z_1, z_2), that triple and $(f(z), f(z_1), f(z_2))$ have the same orientation; it is said to be *opposite* if, for each ordered noncollinear triple (z, z_1, z_2), that triple and $(f(z), f(z_1), f(z_2))$ have different orientations. This definition is bases on the assumption that, if z_0, z_1 and z_2 are three noncollinear points and f is a similarity, then $(f(z), f(z_1), f(z_2))$ are also noncollinear. This is not hard to show and is left as:

EXERCISE 5.16. *Let z_0, z_1 and z_2 be three noncollinear points in the complex plane and let f be any similarity. Prove that $(f(z), f(z_1), f(z_2))$ are also noncollinear.*

It is not at all obvious that a similarity must either preserve the orientation of all noncollinear triples or reverse the orientation of all noncollinear triples. We could prove: if f is a similarity of euclidean plane, then f is either direct or opposite. However, we choose to avoid dealing with this result directly since it will follow as a simple corollary of our classification of the similarities of the euclidean plane.

Let (z, z_1, z_2) be any noncollinear triple and let b be any complex number. Clearly, the translation of the plane obtained by adding b to each complex number does not change the orientation of any noncollinear triple, i.e. (z, z_1, z_2) and $(z+b, z_1+b, z_2+b)$ have the same orientation. Let m be any positive real number. Clearly, the dilation of the plane obtained by multiplying each complex number by m does not change the orientation of any noncollinear triple, i.e. (z, z_1, z_2) and (mz, mz_1, mz_2) have the same orientation. Finally, let u be any complex number of unit length. Clearly, the rotation of the plane obtained by multiplying each complex number by u does not change the orientation of any noncollinear triple, i.e. (z, z_1, z_2) and (uz, uz_1, uz_2) have the same orientation. Combining, these three observations, we have the following strengthening of Lemma **??**:

LEMMA 10. *Let $f(z) = az+b$ be any complex linear function. Then f is a direct similarity of the complex plane with magnification $|a|$.*

We may also prove an extension of Lemma **??** from Chapter 2.

LEMMA 11. *Let z_0, z_1 and z_2 be three noncollinear points in the complex plane and let f and g be two similarities.*

(i) *If $f(z_0) = g(z_0)$, $f(z_1) = g(z_1)$ and $f(z_2) = g(z_2)$, then $f(z) = g(z)$, for all $z \in \mathbb{C}$.*
(ii) *If $f(z_0) = g(z_0)$ and $f(z_1) = g(z_1)$ and f and g are both direct or both opposite, then $f(z) = g(z)$, for all $z \in \mathbb{C}$.*

PROOF. We start by reducing case (ii) to case (i). Let $z_i' = f(z_i) = g(z_i)$, for $i = 0, 1$. Then $m = \frac{|z_1'-z_0'|}{z_1-z_0}$ the magnification of both f and g. It follows that $|f(z_2) - z_i'| = m|z_2 - z_i|$, for $i = 0, 1$. It follows that $z_0', z_1', f(z_2)$ are the vertices of a triangle similar to the triangle with vertices z_0, z_1 and z_2. Similarly, $z_0', z_1', g(z_2)$ are the vertices of a triangle similar to the triangle with vertices z_0, z_1 and z_2. There are just two such triangles with the segment joining z_0' and z_1' as a side: we get one when z_0', z_1' and $f(z_2)$ $(g(z_2))$ are clockwise orientated and the other when z_0', z_1' and $f(z_2)$ $(g(z_2))$ are counterclockwise orientated. If

f and g are both direct or both opposite, then z_0', z_1' and $f(z_2)$ and z_0', z_1' and $g(z_2)$ have the same orientation and $f(z_2) = g(z_2)$.

Now let $z_2' f(z_2) = g(z_2)$ and let z be any point. Since z_0, z_1 and z_2 are noncollinear, z is one of the three lines through pairs of these points. Hence, without loss of generality, we may assume that z_0, z_1 and z are noncollinear. As above, z_0', z_1' and $f(z)$ $(g(z))$ are the vertices of a triangle similar to the triangle with vertices z_0, z_1 and z. Again there are just two such triangles with the segment joining z_0' and z_1' as a side. Let v and w be the third vertices for these two triangles. Hence $f(z)$ must be one of these and so must $g(z)$. Note that the points equidistant from v and w are the points on the line through z_0' and z_1' and that z_2' is not on this line. Hence, $|z_2' - v| \neq |z_2' - w|$. But $|z_2' - f(z_2)| = |z_2' - g(z_2)|$. Hence either $f(z) = v$ and $g(z) = v$ or $f(z) = w$ and $g(z) = w$. In either case, $f(z) = g(z)$. □

THEOREM 8. *[Theorem **??**] If f is a direct similarity of the complex plane, then f is a linear function. If $f(z) = az + b$ is any linear function, then f is a direct similarity of the complex plane. The magnification of this similarity is $|a|$ and f is a direct congruence if and only if $|a| = 1$.*

PROOF. We have just shown that, if f is a linear function, then it is a direct similarity. Now assume that f is a direct similarity with magnification m; we must show that f is given by a linear function. If we can construct a direct linear function g which agrees with f at two distinct points, then, by Lemma **??** (ii), we can conclude that $f = g$, i.e. f is the linear function g. Let $b = f(0)$ and let $a = f(1) - f(0)$. Since f is a similarity with magnification m,

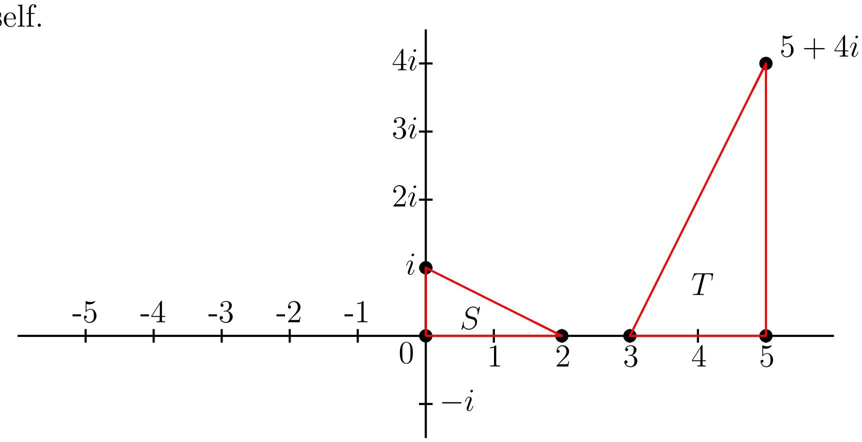
$$|a| = |f(1) - f(0)| = m|1 - 0| = m.$$

Let g be the linear function $g(z) = az + b$. Then g and f are two direct similarities such that $g(0) = b = f(0)$ and

$$g(1) = a(1) + b = (f(1) - f(0))(1) + f(0) = f(1);$$

and are, therefore, equal. □

This theorem gives us a powerful algebraic tool for the study of direct similarities. For example, consider the two triangles pictured below. They are clearly similar and the corresponding vertices have the same orientation. We conclude that there is a direct similarity (a linear function) which maps one triangle onto the other. Let f be the similarity which maps triangle S onto triangle T. We see that f must map i to 3, 0 to 5 and 2 to $5 + 4i$. We conclude that f preserves the orientation of the vertices of S and therefore may be given by a linear function. If we let $f(z) = az + b$ we have $5 = f(0) = a(0) + b$ and $3 = f(i) = a(i) + b$. Solving these two equations give $a = 2i$, $b = 5$ and $f(z) = 2iz + 5$. We can verify that this is indeed the correct similarity by checking how it maps the third vertex:$f(2) = 2i(2) + 5 = 5 + 4i$. We will return to this example in the next section after we investigate in greater detail just how linear functions map the complex plain onto itself.



Exercise 5.17. *Find the equation of the similarity which maps triangle T onto triangle S.*

5.4. Inverses and Fixed Points

In general, the inverse of the linear function $f(z) = az + b$ is the linear function $g(z) = \frac{1}{a}z - \frac{b}{a}$. This can be verified by direct computation:

$$f(g(z)) = a(\frac{1}{a}z - \frac{b}{a}) + b = z; \quad \text{and} \quad g(f(z)) = \frac{1}{a}(az + b) - \frac{b}{a} = z.$$

As we showed in the previous chapter, $\frac{1}{a} = \frac{\bar{a}}{|a|^2}$. Hence the inverse of f can be written in the form $g(z) = \frac{\bar{a}}{|a|^2}z - \frac{b\bar{a}}{|a|^2}$. We have proved:

Lemma 12. *The linear function $f(z) = az+b$ has the linear function $g(z) = \frac{\bar{a}}{|a|^2}z - \frac{b\bar{a}}{|a|^2}$ as its inverse.*

This lemma give us the following corollary to Theorem **??**:

Corollary 8.1. *Let f be a direct similarity of the euclidean line. Then:*

(i) *f has an inverse g which is also a direct similarity; furthermore, the magnification of g is the reciprocal of the magnification of f.*
(ii) *f is a congruence if and only if its inverse is a congruence.*

We can apply this lemma and corollary to the example discussed at the end of the last section. Recall that the similarity that mapped triangle S onto triangle T was given by the linear function $f(z) = 2iz + 5$. By the lemma, the inverse of f is $g(z) = \frac{\overline{2i}}{|2i|^2}z - \frac{5\overline{2i}}{|2i|^2}$ which simplifies to $g(z) = -\frac{i}{2}z + \frac{5i}{2}$. One easily checks that g maps triangle T onto triangle S. This should confirm your conclusions in Exercise **??**. We may also check that the magnification of f is 2 ($\sqrt{2i\overline{2i}} = \sqrt{-4i^2} = 2$) and that the magnification of g is $\frac{1}{2}$ ($\sqrt{\frac{i}{2}\overline{\frac{i}{2}}} = \sqrt{-\frac{i^2}{4}} = \frac{1}{2}$).

We continue to duplicate, for the complex linear functions, the results about real linear functions from the first chapter. Compare the next lemma and its proof with Lemma **??** and its proof.

LEMMA 13. *[Lemma **??**] Let $f(z) = az + b$.*

(i) *If $a = 1$ and $b = 0$, then f is the identity function.*
(ii) *If $a = 1$ and $b \neq 0$, then f has no fixed points.*
(iii) *If $a \neq 1$, then $\frac{b}{1-a}$ is the unique fixed point of f.*

PROOF. The complex number z is a fixed point for f if and only if $z = az + b$, i.e., if and only if $(1 - a)z = b$.

(i) If $a = 1$ and $b = 0$, $(1 - a)z = b$ for all z and f is the identity

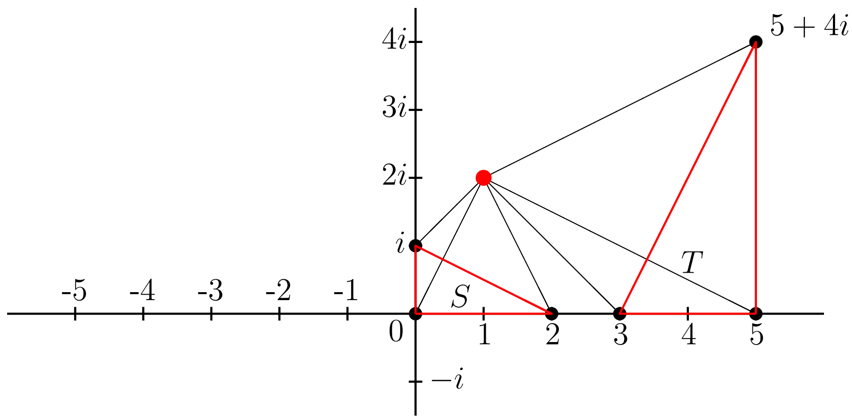
(ii) If $a = 1$ and $b \neq 0$, $(1 - a)z = b$ for no z and f has no fixed point.

(iii) If $a \neq 1$, $(1 - a)z = b$ if and only if $z = \frac{b}{1-a}$ and $\frac{b}{1-a}$ is the unique fixed point of f. □

Returning to our example, $f(z) = 2iz + 5$, we see that the fixed point is:

$$\frac{5}{1-2i} = \frac{5(1+2i)}{(1-2i)(1+2i)} = \frac{5(1+2i)}{5} = 1 + 2i.$$

One easily verifies that $f(1 + 2i) = 2i(1 + 2i) + 5 = 2i - 4 + 5 = 1 + 2i$. We may now give a very nice geometric description of this similarity: the plane is rotated counterclockwise by 90deg about $1 + 2i$ and simultaneously dilated from that point by a factor of 2. In the following figure, you can see that the segment joining the center to the vertex 2 (of S) is rotated 90deg and doubled in length to get the segment joining the center and the vertex $5 + 4i$ (of T). One can easily check the other vertices are mapped in the same manner.



EXERCISE 5.18. *Find the fixed points for each of the the following similarities and give a descriptions for each of these similarities:*

(i) $f(z) = iz + 2$;
(ii) $g(z) = -z + 2$
(iii) $h(z) = 2z - 1$
(iv) $k(z) = 2iz + 1$

EXERCISE 5.19. *Show that a direct similarity and its inverse have the same fixed point.*

5.5. Direct Congruences and Similarities

Consider the direct similarity of the plane given by the linear function $f(z) = az + b$. If $|a| = 1$, we have a congruence; if $a = 1$ we have the identity or a direct similarity without fixed point. Assume that $a = 1$, that is $f(z) = z + b$. We have already noted that this is the translation which maps 0 to b. We will use the same notation as in the real case: we denote this similarity by $t_{[b]}$ and call it the *translation* by b.

Next lets consider the analogue of the (real) reflection $f(x) = -x + b$. The complex function $f(z) = -z + b$, as we have just seen, has $\frac{b}{2}$ as fixed point and, in fact, "reflects" each point through $\frac{b}{2}$. To see this, we simply compute the mid point of the segment between z and $f(z)$:

$$\frac{z + f(z)}{2} = \frac{z - z + b}{2} = \frac{b}{2}.$$

One may also describe f as the 180 deg rotation or *half-turn about* $\frac{b}{2}$. Since this transformation preserves orientation, we will use the term half-turn and leave the term "reflection" for reflections through lines - which do reverse orientation.

The translations and half-turns are just two of many special types of direct similarities. Rather than to continue to study these special types one at a time, we will move directly to an investigation of the geometry of the general linear function. Let $f(z) = az + b$ be a fixed linear function and let $a = m(\cos(\theta) + i\sin(\theta))$. Assume further that f is not a translation ($a \neq 1$) and, therefore, that $c = \frac{b}{1-a}$ is its fixed point. Let v and w be two points in the plane, consider the segment that they define and consider the image of this segment under f. In proving that f was a similarity we proved that all lengths were stretched (or shrunk) by the same factor, $|a| = m$, the magnification of the similarity. So $|f(v) - f(w)|$, the length of image of the segment, is $m|v - w|$.

Another important property of a segment is its "orientation" or the angle it makes with the "horizontal". To make this concept of orientation precise, recall that the segment from v to w is parallel to the segment joining 0 to $w - v$. We may write $w - v$ in polar form: $w - v = m'(\cos(\phi) + i\sin(\phi))$. In this form m' is the length of the segment and ϕ is its orientation, i.e., the angle the parallel segment joining 0 to $w - v$ makes with the positive real axis. Next, we write $f(w) - f(v)$ in polar coordinates:

$$f(w)-f(v) = aw+b-(av+b) = a(w-v) = mm'(\cos(\theta+\phi)+i\sin(\theta+\phi)).$$

It follows that the segment joining v to w has its length stretched by the factor m' and its orientation rotated counterclockwise by the angle of measure θ. Now, if we take v to be the fixed point c, we see that an arbitrary point w is rotated counterclockwise by an angle of measure θ about c and "dilated by the factor m, again from the point c. Thus we call f a *dilating-rotation* and we denote f by $r_{[m,\theta,c]}$. We have:

$$r_{[m,\theta,c]}(z) = az + (1 - a)c, \quad \text{where} \quad a = m(\cos(\theta) + i\sin(\theta)).$$

There are two cases which are of special interest:

- If $\theta = 0$, then there is no rotation and $r_{[m,0,c]}$ is the *dilation* with magnification m and center c.
- If $m = 1$, then there is no change in lengths and $r_{[1,\theta,c]}$ is the *rotation* counterclockwise by an angle of measure θ about c.

We may summarize our discussion in two corollaries to Theorem **??**

COROLLARY 8.2. *The only direct congruences of the euclidean plane are the translations and the rotations.*

COROLLARY 8.3. *The only direct similarities of the euclidean plane which are not congruences are the dilations and dilating-rotations.*

We close this chapter with a collection of exercises.

EXERCISE 5.20. *Let f denote the rotation by 90 degrees about* 1*, let g denote the rotation by 90 degrees about i and let t denote the translation which takes* 0 *to the point* $1+i$*.*

(i) *Give the formula for each of these linear functions.*
(ii) *Both algebraically and geometrically, give the inverse for each of these linear functions.*
(iii) *Both algebraically and geometrically, give the composition, in each order, for each pair of these linear functions.*

EXERCISE 5.21. *Let f denote the half-turn (rotation by 180 degrees) about* 1*, let g denote the rotation by 90 degrees about $-i$ and let t denote the translation which takes* 0 *to the point* $1-i$*.*

(i) *Give the formula for each of these linear functions.*
(ii) *Both algebraically and geometrically, give the inverse for each of these linear functions.*
(iii) *Both algebraically and geometrically, give the composition, in each order, for each pair of these linear functions.*

EXERCISE 5.22.
(i) *Explain geometrically the result of composing two half-turns.*
(ii) *Explain geometrically the result of composing a translation and a half-turn.*

EXERCISE 5.23. *Consider the following geometry game called "leap frog" which is played as follows.*

- *Select n points v_1, $v_2, \cdots$, v_n, at random in the plane to form the "course".*
- *Select a starting point z_0 and leap frog through the course: construct z_1 so that v_1 is the midpoint of the segment joining z_0 and z_1; construct z_2 so that v_2 is the midpoint of the segment joining z_1 and z_2; $\cdots$; construct z_n so that v_n is the midpoint of the segment joining z_{n-1} and z_n.*
- *From the same starting point z_0 leap frog through the course again but in the reverse order: construct z_2 so that v_n is the midpoint of the segment joining $z_0' = z_0$ and z_2; construct z_2' so that v_{n-1} is the midpoint of the segment joining z_2 and z_2'; etc.*

If you play this game with a variety of courses, you should discover that, when n is odd, the direction in which you traverse the course makes no difference, i.e., $z_n' = z_n$ for all starting points z_0! You should also discover that, when n is even, z_n' and z_n are almost certain to be different; the only exceptions occur when the course has a very "nonrandom" structure, like the vertices of a square. Explain this phenomena!

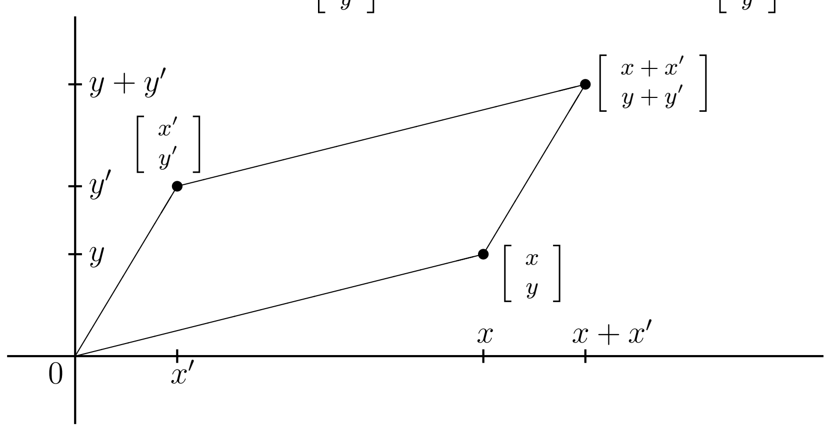
What about the opposite similarities? They correspond to the *conjugate linear functions,* functions of the form $f(z) = a\overline{z} + b$. We could could carry out a parallel development for these functions. We elect instead to move on the development of a more general setting that encompasses both direct and opposite similarities and much more: the matrix linear function. However, the development of the theory of conjugate linear functions will make an interesting and challenging research project for the interested reader.

CHAPTER 6

The Matrix Linear Function

6.1. Vectors and the Coordinate Plane

Instead of dealing with a number system like the real or complex numbers, we turn our attention to the coordinatized plane, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. The points are represented by ordered pairs of real numbers and we think of them as they are normally arranged in the coordinate plane: the x-axis is a horizontal line and the y-axis a vertical line intersecting at the origin. Instead of (x, y), the usual method of writing coordinates, we write our coordinates in the form of a column vector: $\begin{bmatrix} x \\ y \end{bmatrix}$. While not a number system like the real or complex numbers, these column vectors do have some number like structure. In particular, they admit a natural, component-wise addition: for any two vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x' \\ y' \end{bmatrix}$, we define their sum by $\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + x' \\ y + y' \end{bmatrix}$. We also define multiplication of a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ by a real number r by $r\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} rx \\ ry \end{bmatrix}$.



As in the case of complex numbers, we are going to want to "do algebra" with these vectors. Therefore we need a clear understanding of the rules for vector algebra. These rules are very natural and easy to verify:

Lemma 14. *For all* $\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x' \\ y' \end{bmatrix}, \begin{bmatrix} x'' \\ y'' \end{bmatrix} \in \mathbb{R}^2$ *and all* $r, r' \in \mathbb{R}$*:*

(i) *(Associativity of +)*
$$\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x' \\ y' \end{bmatrix}\right) + \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \left(\begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} x'' \\ y'' \end{bmatrix}\right);$$

(ii) *(Commutativity of +)* $\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix};$

(iii) *(Additive identity,* $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$*)*
$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix};$$

(iv) *(Additive inverse,* $\begin{bmatrix} -x \\ -y \end{bmatrix}$*)*
$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -x \\ -y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix};$$

(v) *(Associativity of scalar multiplication)*
$$(r \times r')\begin{bmatrix} x \\ y \end{bmatrix} = r \times \left(r'\begin{bmatrix} x \\ y \end{bmatrix}\right);$$

(vi) *(Multiplication by 0)* $0\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix};$

(vii) *(Multiplication by 1)* $1\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix};$

(viii) *(Distributivity of scalar multiplication over addition)*
$$(r + r')\begin{bmatrix} x \\ y \end{bmatrix} = r\begin{bmatrix} x \\ y \end{bmatrix} + r'\begin{bmatrix} x \\ y \end{bmatrix} \text{ and}$$
$$r\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x' \\ y' \end{bmatrix}\right) = r\begin{bmatrix} x \\ y \end{bmatrix} + r\begin{bmatrix} x' \\ y' \end{bmatrix};$$

Exercise 6.1. *Prove this lemma.*

While our main concern will be 2-dimensional geometry, this vector approach can be used to model 3-dimensional geometry as well, in fact, n-dimensional geometry for any n. In view of our main goal, we will include the generalizations to higher dimensions as a series of exercises that can be skipped without detracting from our development. To make it clear that they may be skipped, these exercises will be denoted by [$3D$].

Exercise 6.2. [$3D$] *We define addition and scalar multiplication of 3-dimensional vectors by:*
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x + x' \\ y + y' \\ z + z' \end{bmatrix} \text{ and } r\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} rx \\ ry \\ rz \end{bmatrix}.$$
State and prove the 3-dimensional analog to Lemma **??**.

We are going to use the coordinate plane as a model for euclidean geometry. The vectors are the points. There are two natural ways to describe lines. One is given in terms of a point on the line and the direction of the line and the other is given in terms of two points on the line. We start with the simplest types of lines to describe, the lines through the origin $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The points on such a line are simply the scalar multiples of a *nonzero* vector:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} v \\ w \end{bmatrix}, \text{ for all } \lambda.$$

If $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is any point, we may add $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ to each point on the above line through the origin to get an equation for the line through $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and parallel to the original line through the origin:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} v \\ w \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \text{ for all real numbers } \lambda.$$

This is called the *point - direction-vector* equation for a line.

EXERCISE 6.3. *Give the point - direction-vector equation of the line through* $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$ *with direction vector* $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and the point - direction-vector equation of the line through* $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ *with this same direction-vector.*

(i) *Draw these two lines in the plane and observe that they are parallel.*
(ii) *Show algebraically that two lines with the same direction-vector are either parallel or equal.*

The point - direction-vector representation of a lines in the coordinatized plane is slightly more general than the usual slope - intercept representation in that vertical lines have a point - direction-vector representation (the direction-vector has the form $\begin{bmatrix} 0 \\ w \end{bmatrix}$, $w \neq 0$) but not a slope - intercept representation.

EXERCISE 6.4. *Consider the line with equation* $y = -2x + 5$

(i) *Give a point-direction-vector equation for this line;*
(ii) *Give the point-direction-vector equation for the parallel line through* $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$*;*
(iii) *Give the traditional equation for the parallel line through* $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$*;*

EXERCISE 6.5. *Demonstrate the general relationship between point - direction-vector representation (when* $v \neq 0$*) and the slope - intercept representation:*

(i) *Replace the equation* $\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} v \\ w \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ *by two equations, one for* x *and one for* y*. Then eliminate* λ *to get the slope - intercept representation for this line.*

(ii) *Show that two lines with direction-vectors that are scalar multiples of one another have the same slope.*

(iii) *Find a point - direction-vector representation for the line with slope - intercept representation $y = ax + b$. Note that there are very many choices.*

EXERCISE 6.6. [3*D*] *Replacing the 2D vectors by 3D vectors in the above definition give the lines in 3-space. Consider the lines: through* $\begin{bmatrix} -4 \\ 1 \\ 2 \end{bmatrix}$ *with direction-vector* $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *and through* $\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ *with direction-vector* $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

(i) *Compute the equations for these two lines.*

(ii) *Show that these two lines never meet.*

Note that the second line lies entirely in the x, y-plane but that the first line is tilted with respect to this plane. So these lines are not parallel - they are skew. In 3-space two lines are defined to be parallel if the direction-vectors of the two lines are scalar multiple of one another.

(iii) *Find the equation through* $\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ *parallel to the first line.*

Now let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ be any two distinct points. We may get an equation for the line through these two points by taking $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ as the direction-vector and $\begin{bmatrix} x_0 \\ y_) \end{bmatrix}$ as the point in the point - direction-vector equation of a line, giving:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \text{ for all real numbers } \lambda.$$

Note that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, when $\lambda = 0$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, when $\lambda = 1$. So this is indeed the line through $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$. Reorganizing this this equation we get the standard *affine* equation for a line:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + (1 - \lambda) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \text{ for all real numbers } \lambda.$$

This is also called the *two-point* equation for the line.

The interesting and useful fact about this last equation for a line is that it relates the "internal" and "external" coordinates for the points on the line: Let l denote the line through $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ Now, let λ

denote the 1-dimensional coordinate system on the line which assigns the coordinate 0 to $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and 1 to $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$. If Q is the point on l with coordinate λ, then the coordinates of Q in the plane are given by the above equation. For example, $\lambda = \frac{1}{2}$ is the "internal" coordinate of the midpoint of the segment joining $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and

$$\begin{bmatrix} \frac{x_1+x_0}{2} \\ \frac{y_1+y_0}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + (1-\frac{1}{2})\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

are the "external" coordinates of this midpoint.

We define the distance between two points $P_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and $P_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ in the usual way:

$$|P_0P_1| = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}.$$

Thinking of the point $P = \begin{bmatrix} x \\ y \end{bmatrix}$ as a vector, its length simply its distance from the origin: $|P| = \sqrt{x^2 + y^2}$.

EXERCISE 6.7. *Verify that* $P_{\frac{1}{2}} = \begin{bmatrix} \frac{x_1+x_0}{2} \\ \frac{y_1+y_0}{2} \end{bmatrix}$ *is indeed the midpoint of the segment joining* P_0 *and* P_1. *That is, show that* $|P_0P_{\frac{1}{2}}| = |P_{\frac{1}{2}}P_1|$.

EXERCISE 6.8. *Consider the triangle with vertices* $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$.

(i) *Compute an equation for each of the medians of this triangle.*
(ii) *Show that these medians have a common point of intersection (called the centroid of the triangle).*

EXERCISE 6.9. [3*D*] *Consider the triangle with vertices*
$\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}$.

(i) *Compute an equation for each of the medians of this triangle.*
(ii) *Compute the coordinates of the centroid of this triangle.*

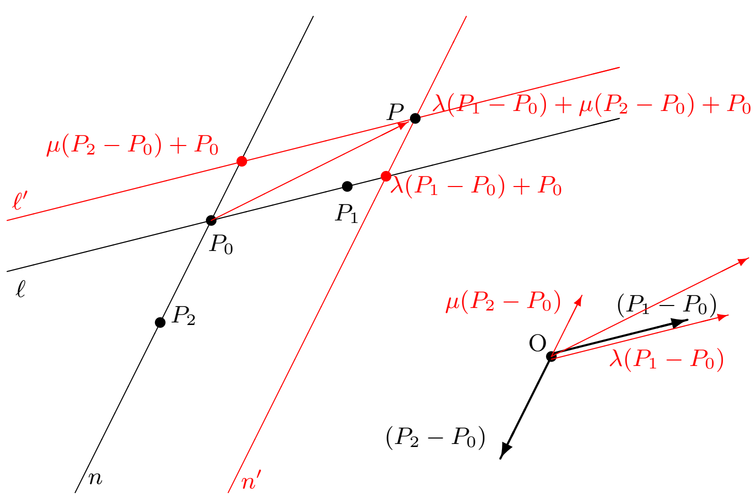
EXERCISE 6.10. *Consider the triangle with vertices*
$\begin{bmatrix} a \\ b \end{bmatrix}$, $\begin{bmatrix} c \\ d \end{bmatrix}$ *and* $\begin{bmatrix} e \\ f \end{bmatrix}$.
Show that the medians have a common point of intersection, the centroid, and verify that it is the point $\begin{bmatrix} \frac{a+c+e}{3} \\ \frac{b+d+f}{3} \end{bmatrix}$

Now consider two distinct points in the plane $P_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $P_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and the line that they determin. As we noted, the parametric equation for the line $P(\lambda) = \lambda P_1 + (1-\lambda)P_0 = \lambda(P_1 - P_0) + P_0$ may be thought of as coordiaatizing the line by λ. In this coordinate system, P_0 has the coordinate 0 and P_1 has the coordinate 1. The distance between two points $P(\lambda)$ and $P(\mu)$, in this coordinate system, is simply $|\lambda - \mu|$. How does the distance between $P(\lambda)$ and $P(\mu)$ in this coordinate system for the line compare with the distance between $P(\lambda)$ and $P(\mu)$ in the plane? We answer this question in the next Lemma.

LEMMA 15. *Let $P(\lambda) = \lambda P_1 + (1-\lambda)P_0$ be the parameterization of the line ℓ given by the points P_1 and P_0 and let $P(\lambda)$ and $P(\mu)$ be two points on this line. Then $|P(\lambda)P(\mu)| = |\lambda - \mu||P_1P_0|$. In particular, if $|P_1P_0| = 1$, the distance between any two points on ℓ, in the coordinate system given by P_1 and P_0, equals the distance between those points in the plane.*

PROOF. By direct computation: $|P(\lambda)P(\mu)| =$
$$= \sqrt{((\lambda x_1 + (1-\lambda)x_0) - (\mu x_1 + (1-\mu)x_0))^2 + ((\lambda y_1 + (1-\lambda)y_0) - (\mu y_1 + (1-\mu)y_0))^2}$$
$$= \sqrt{((\lambda - \mu)(x_1 - x_0))^2 + ((\lambda - \mu)(y_1 - y_0))^2}$$
$$= \sqrt{(\lambda - \mu)^2}\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} = |\lambda - \mu||P_1P_0|.$$
□

With three non-collinear points we may coordinate the entire plane. Let P_0, P_1 and P_2 be three non-collinear points. Let ℓ be the line through P_0 and P_1 and n be the line through P_0 and P_2. Since the points are non-collinear, the lines are distinct, intersection only at P_0. See the following figure:



Let P be any point in the plane. If P is on ℓ, it can be written in the form $\lambda(P_1 - P_0) + P_0$ and, if it is on n, it can be written in the form

$\mu(P_2-P_0)+P_0$. Suppose P is not on either of these lines. Let ℓ' be the line through P parallel to ℓ; it intersects n a point that can be written in the form $\mu(P_2-P_0)+P_0$. Similarly n', the line through P parallel to n, intersects ℓ in a point that can be written as $\lambda(P_1-P_0)+P_0$. It follows easily that P can be written in the form $\lambda(P_1-P_0)+\mu(P_2-P_0)+P_0$ and that it is uniquely determined by its *coordinates* (λ, μ).

6.2. Matrices

We need to introduce one more algebraic concept before we can finally define a matrix linear functions; that is, of course, the concept of a matrix. By an $n \times m$ (n by m) *matrix*, we mean a rectangular array of nm real numbers arranged in n rows and m columns. Our 2-dimensional vectors are 2×1 matrices and n-dimensional vectors are $n \times 1$ matrices. Beyond vectors, we will mainly be interested in 2×2 matrices. For example:

$$\begin{bmatrix} 4 & 1 \\ -\frac{1}{2} & -2 \end{bmatrix}, \quad \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Matrices may be multiplied by scalars and matrices of the same dimensions can be added component wise the rules for these operations are just slight generalizations of the rules we have already developed for vectors. We list these rules here for 2×2 matrices and we leave it as an exercise to state and prove the corresponding results for $n \times m$ matrices.

LEMMA 16. *For all* $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix}, \begin{bmatrix} m''_{11} & m''_{12} \\ m''_{21} & m''_{22} \end{bmatrix}$ *and all* $r, r' \in \mathbb{R}$*:*

(i) *(Associativity of +)*
$$\left(\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + \begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix}\right) + \begin{bmatrix} m''_{11} & m''_{12} \\ m''_{21} & m''_{22} \end{bmatrix}$$
$$= \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + \left(\begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix} + \begin{bmatrix} m''_{11} & m''_{12} \\ m''_{21} & m''_{22} \end{bmatrix}\right);$$

(ii) *(Commutativity of +)*
$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + \begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix} = \begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix} + \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix};$$

(iii) *(Additive identity,* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$*)*
$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix};$$

(iv) *(Additive inverse,* $\begin{bmatrix} -m_{11} & -m_{12} \\ -m_{21} & -m_{22} \end{bmatrix}$*)*
$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + \begin{bmatrix} -m_{11} & -m_{12} \\ -m_{21} & -m_{22} \end{bmatrix} = \begin{bmatrix} -m_{11} & -m_{12} \\ -m_{21} & -m_{22} \end{bmatrix} + \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix};$$

(v) *(Associativity of scalar multiplication)*

$$(r \times r')\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = r \times \left(r'\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\right);$$

(vi) *(Multiplication by* 0*)* $0\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$;

(vii) *(Multiplication by* 1*)* $1\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$;

(viii) *(Distributivity)*

$$(r + r')\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = r\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + r'\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \text{ and}$$

$$r\left(\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\right) = r\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + r\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}.$$

EXERCISE 6.11. *Complete at least* (i) *below:*

(i) *Prove Lemma* **??**.

(ii) [3D] *Define addition and scalar multiplication of* 3×3*-matrices and state and prove the 3-dimensional analog to Lemma* **??**.

(iii) *Define addition and scalar multiplication of* $n \times m$*-matrices and state and prove this most general analog to Lemma* **??**.

The last operation we must introduce is matrix multiplication. We will want to multiply a 2×2 matrix times a 2-dimensional vector and we will want to multiply a 2×2 matrix times another 2×2 matrix. The best way to define these operations is to define the very general operation of multiplying an $n \times m$-matrix times an $m \times k$. The key requirement in multiplying matrices is that the number of columns of the first matrix equals the number of rows of the second matrix:

$$\begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{i1} & \dots & a_{im} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1k} \\ \vdots & & \vdots & & \vdots \\ b_{m1} & \dots & b_{mj} & \dots & b_{mk} \end{bmatrix} = \begin{bmatrix} c_{11} & \dots & c_{1j} & \dots & c_{1k} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \dots & c_{ij} & \dots & c_{ik} \\ \vdots & & \vdots & & \vdots \\ c_{n1} & \dots & c_{nj} & \dots & c_{nk} \end{bmatrix}$$

The resulting matrix has the same number of rows as the first matrix and the same number of columns as the second matrix. An arbitrary entry in the product, c_{ij}, is defined as follows: locate the ith row of the first matrix and the jth column of the second matrix and observe that they each consist of a (vertical or horizontal) string of m numbers - we simply multiply corresponding numbers together and add up these products:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{im}b_{mj}.$$

Specifically:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m_{11}x + m_{12}y \\ m_{21}x + m_{22}y \end{bmatrix} \quad \textit{and}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix}$$

EXERCISE 6.12. *Carry out the following multiplications:*

(i) $\begin{bmatrix} 4 & 4 \\ \frac{1}{2} & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -\frac{3}{2} \end{bmatrix}$

(ii) $\begin{bmatrix} 4 & 4 \\ \frac{1}{2} & -1 \end{bmatrix}\begin{bmatrix} 3 & -6 \\ -\frac{3}{2} & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & -6 \\ -\frac{3}{2} & 0 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ \frac{1}{2} & -1 \end{bmatrix}$

(iv) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$

(v) $\begin{bmatrix} 3 & -6 & 1 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 4 & 0 & -7 \\ \frac{1}{2} & -1 & -1 & 0 \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 \end{bmatrix}$

Our final set of algebraic rules will be for matrix multiplication. And here there is a slight but very important deviation from all of the rule sets we have discussed above. As you observed in the last exercise [(ii) & (iii)], *multiplication of matrices is not commutative!* As usual, we will state these rules for the 2-dimensional case and give the reader the option of giving the proof for that case or stating and proving the general results.

LEMMA 17. *For all* $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix}, \begin{bmatrix} m''_{11} & m''_{12} \\ m''_{21} & m''_{22} \end{bmatrix}$ *and all* $\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x' \\ y' \end{bmatrix}$*:*

(i) *(Associativity)*

$$\left(\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix}\right)\begin{bmatrix} m''_{11} & m''_{12} \\ m''_{21} & m''_{22} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\left(\begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix}\begin{bmatrix} m''_{11} & m''_{12} \\ m''_{21} & m''_{22} \end{bmatrix}\right) \text{ and}$$

$$\left(\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\left(\begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right);$$

(ii) *(Multiplicative identity,* $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$*)*

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix};$$

(iii) *(Multiplicative inverses) When* $\Delta = m_{11}m_{22} - m_{12}m_{21} \neq 0$, *then* $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} \frac{m_{22}}{\Delta} & -\frac{m_{21}}{\Delta} \\ -\frac{m_{12}}{\Delta} & \frac{m_{11}}{\Delta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and*

$$\begin{bmatrix} \frac{m_{22}}{\Delta} & -\frac{m_{21}}{\Delta} \\ -\frac{m_{12}}{\Delta} & \frac{m_{11}}{\Delta} \end{bmatrix}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

(iv) *(Distributivity)*

$$\left(\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + \begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix}\right)\begin{bmatrix} m''_{11} & m''_{12} \\ m''_{21} & m''_{22} \end{bmatrix}$$

$$= \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} m''_{11} & m''_{12} \\ m''_{21} & m''_{22} \end{bmatrix} + \begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix} \begin{bmatrix} m''_{11} & m''_{12} \\ m''_{21} & m''_{22} \end{bmatrix} \textit{ and}$$
$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \left(\begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix} + \begin{bmatrix} m''_{11} & m''_{12} \\ m''_{21} & m''_{22} \end{bmatrix} \right)$$
$$= \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix} + \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} m''_{11} & m''_{12} \\ m''_{21} & m''_{22} \end{bmatrix} \textit{ and}$$
$$\left(\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + \begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \textit{ and}$$
$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x' \\ y' \end{bmatrix} \right)$$
$$= \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

In manipulating matrix expressions, it is frequently convenient to represent matrices by a single letter capital letter. We usually denote the multiplicative identity by I. When a matrix M has an inverse, we usually denote that inverse by M^{-1}. M^{-1} exists, Rule (iii) can then be stated the in compact form:

M^{-1} exists whenever $\Delta(M) \neq 0$ and then $MM^{-1} = M^{-1}M = I$, where $\Delta(M) = m_{11}m_{22} - m_{12}m_{21}$.

$\Delta(M)$ is called the *determent* of M.

EXERCISE 6.13. *Prove Lemma* **??**.

EXERCISE 6.14. [3*D*] *State and prove the 3-dimensional analog to Lemma* **??**. *The only difficult part is determining when a* 3×3 *matrix has a multiplicative inverse. So for that part, you may wish to wait until we discuss the determent in detail.*

We say that matrix multiplication is not commutative. This does not mean that no pair of two $n \times n$ matrices commute. It simply means that any given pair of two $n \times n$ matrices *may* not commute. What this means in terms of our ability to carry out algebraic computations with matrices is that we must simply be very careful not to change order of matrices being multiplied unless we know for a fact that those two matrices commute. In particular, we know that the identity commutes with every matrix and that a matrix and its inverse (if that exists) commute.

Taking a closer look at Lemma **??** (iii), all that is claimed is that, when $\Delta = m_{11}m_{22} - m_{12}m_{21} \neq 0$, $\begin{bmatrix} \frac{m_{22}}{\Delta} & -\frac{m_{21}}{\Delta} \\ -\frac{m_{12}}{\Delta} & \frac{m_{11}}{\Delta} \end{bmatrix}$ is *an* inverse for $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$. A much stronger statement is true:

LEMMA 18. *Consider the* 2×2 *matrix* $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$.

(i) *M has an inverse if and only if* $\Delta = m_{11}m_{22} - m_{12}m_{21} \neq 0$.
(ii) *If M has an inverse, the inverse is unique.*

EXERCISE 6.15. *Prove Lemma ??: Convert the matrix equation* $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *as a system of 4 equations in 4 variables and solve.*

EXERCISE 6.16. *Check whether or not each of the following matrices has an inverse.*

$$M = \begin{bmatrix} 4 & 4 \\ \frac{1}{2} & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 3 & -1 \\ -2 & \frac{2}{3} \end{bmatrix}$$

(i) *Compute the inverse for each matrix that has an inverse and check your answer by multiplying that matrix by its inverse.*
(ii) *Compute* $(MH)^{-1}$, $\quad M^{-1}H^{-1}$ *and* $H^{-1}M^{-1}$.
(iii) *Make and verify conjectures about the existence and inverse of the product of two matrices and how it may be computed when it exists.*

Another very useful lemma:

LEMMA 19. *Let M be a 2×2 matrix. Then $\Delta(M) = 0$ if and only if one of the columns of M is a scalar multiple of the other.*

PROOF. Assume that $\begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} = \alpha \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix}$, then
$\Delta(M) = \alpha m_{12}m_{22} - m_{12}\alpha m_{22} = 0$
and the same argument works when the second column is a scalar multiple of the first.

Next, we assume that $\Delta(M) = 0$ and wish to prove that one of the columns of M is a scalar multiple of the other. Here there are several cases to consider. First, we note that if either of the columns of M have only zero entries, it is the 0 multiple of the other. One also easily sees that if either row of M has only zero entries, then one column is a scalar multiple of the other. So we assume that either m_{21} or m_{22} is not zero. Since $m_{11}m_{22} - m_{12}m_{21} = 0$, $m_{11}m_{22} = m_{12}m_{21}$ and either $\begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} = \frac{m_{22}}{m_{21}} \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix}$ or $\begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} = \frac{m_{21}}{m_{22}} \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix}$. □

6.3. The Matrix Linear Function

By a *matrix linear function* we mean a function from the coordinatized plane into the coordinatized plane of the form

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

where $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ is any 2×2 matrix. We often write a matrix linear function in the form $f(X) = MX + B$. One important (and easy to prove) fact about matrix linear functions is:

LEMMA 20. *If $f(X) = MX+B$ and $g(X) = NX+C$ are two matrix linear functions, then their composition is a matrix linear function.*

The proof follows directly from the straight forward matrix computations you will carry out in the next exercise.

EXERCISE 6.17. *Let $f(X) = MX + B$ and $g(X) = NX + C$ be two matrix linear functions, then $f \circ g$ and $g \circ f$ are both matrix linear functions; specifically:*

(i) $f \circ g(X) = (MN)X + (MC + B)$
(ii) $g \circ f(X) + (NM)X + (NB + C)$

Let ℓ be any line in the plane and let P_1 and P_0 be two distinct points on ℓ. Then, as we have seen, $P(\lambda) = \lambda P_1 + (1 - \lambda)P_0$ is a parameterization of ℓ. By direct computation:

$$\begin{aligned} f(P(\lambda)) &= f(\lambda P_1 + (1-\lambda)P_0) \\ &= M(\lambda P_1 + (1-\lambda)P_0) + B \\ &= \lambda MP_! + (1-\lambda)MP_) + (\lambda B + (1-\lambda)B \\ &= \lambda(MP_1 + B) + (1-\lambda)(MP_0 + B) \\ &= \lambda f(P_1) + (1-\lambda)f(P_0). \end{aligned}$$

From this we can draw several conclutions. First, suppose that f restricted to ℓ is not one to one. Then we may choose two distinct points, P_1 and P_0, on ℓ so that $f(P_1) = f(P_0) = Q$. It follows that, in this case, $f(\lambda P_1 + (1-\lambda)P_0) = \lambda Q + (1-\lambda)Q = Q$, for all λ. So $f(\ell)$ is either a single point or f is one to one. In the case that f is one to one, it follows that $f(\ell)$ is the line given parametrically by $\lambda f(P_1) + (1-\lambda)f(P_0)$.

We can actually say more. Fix P_1 and P_0 on ℓ and select any two points $P(\lambda)$ and $P(\mu)$ on ℓ. By Lemma **??**, the distance between $P(\lambda)$ and $P(\mu)$ on the line ℓ and in the plane is $|\lambda - \mu||P_1P_0|$; by the above computation and Lemma **??**, the distance on the line $f(\ell)$ and in the plane between their images under f is $|\lambda - \mu||f(P_1)f(P_0)|$. Hence f restricted to ℓ is a function from the line ℓ to the line $f(\ell)$ that magnifies all distances by the factor of $\frac{|f(P_1)f(P_0)|}{|P_1P_0|}$. We summarize this discussion in the following lemma:

LEMMA 21. *Let $f(X) = MX + B$ be a matrix linear function, let ℓ be any line and let P_0 and P_1 be two distinct points on ℓ. If $f(P_0) = f(P_1)$, all of the points on ℓ map onto $f(P_0) = f(P_1)$. Otherwise, ℓ is mapped onto the line through $f(P_0)$ and $f(P_1)$. Furthermore, f*

restricted to ℓ is a function from the line ℓ to the line $f(\ell)$ that magnifies all distances by the factor of $\frac{|f(P_1)f(P_0)|}{|P_1P_0|}$.

In Chapter one, we considered only functions from a line onto its self and, in that case, we showed that the similarities of the line (thinking of congruences as similarities with magnification 1) were precisely the linear functions. We now turn to a list of the fundamental properties of matrix linear functions:

THEOREM 9. *Let f be a matrix linear function, then the image of f is either a single point, a line or the entire plane. Furthermore, if the image of f is the entire plane then*

(i) *f is one-to-one as well as onto;*
(ii) *f maps lines onto lines;*
(iii) *f maps noncollinear triples onto noncollinear triples.*
(iv) *f maps parallel lines onto parallel lines;*

Finally, if g is a second matrix linear function and P, Q and R a non-collinear triple so that $g(P) = f(P)$, $g(Q) = f(Q)$ and $g(R) = f(R)$, then $g = f$.

PROOF. We write f in the form $f(X) = MX + B$ and consider three cases.
Case 1. M is the matrix of all zeros. In this case, $f(X) = B$ for all points X in the plane.
Case 2. M is not the matrix of all zeros but $\Delta(M) = 0$. By Lemma **??**, M has the form $[C\alpha C]$ (or $[\alpha CC]$), where C is not the zero vector. We have then $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = xC + \alpha yC + B = (x + \alpha y)C + B$. Hence, in this case, the image of f is the line through B and C.
Case 3. $\Delta(M) \neq 0$. By Lemma **??** (iii), M has an inverse M^{-1}. In this case, the matrix linear function $f^{-1}(X) = M^{-1}X - M^{-1}B$ is the inverse of f:

$$f^{-1}(f(X)) = M^{-1}(MX+B)-M^{-1}B = M^{-1}MX+M^{-1}B-M^{-1}B = X$$

(We leave it to the reader to show that $f(f^{-1}(X)) = X$.) We conclude that, in this case, f is one to one and onto; in particular, that the image of f is the entire plane.

Assume that the image of f is the entire plane. Then by Cases 1 and 2 above, Case 3 must hold and $\Delta(M) \neq 0$. Hence f is one to one and onto and (i) is verified. To verify (ii), let ℓ, we simply note that, for any two distinct points P_0 and P_1, $f(P_0)$ and $f(P_1)$ are distinct and apply Lemma **??**. Let P_0, P_1 and P_2 be three non-collinear points. Then P_2 is not on the line through P_0 and P_1. As we have just seen, the line through P_0 and P_1 is mapped *onto* the line through $f(P_0)$ and $f(P_1)$. Since f is one to one, $f(P_2)$ cannot lie on the line through $f(P_0)$ and $f(P_1)$. Thus $f(P_0)$, $f(P_1)$ and $f(P_2)$ are non-collinear. Finally let ℓ and n be two parallel lines. Since f is one to one, $f(P) \neq f(Q)$ for any points $P \in \ell$, $Q \in n$. Hence $f(\ell)$ and $f(n)$ are non-intersecting lines and hence parallel.

Finally, let f and g be matrix linear functions that map the plane onto the plane and let P_0, P_1 and P_2 be a non-collinear triple so that $f(P_0) = g(P_0)$, $f(P_1) = g(P_1)$ and $f(P_2) = g(P_2)$. Let P be any point and write it in terms of the coordinate system given by P_0, P_1 and P_2: $P = \lambda(P_1 - P_0) + \mu(P_2 - P_0) + P_0$. Then by direct computation we have:

$$\begin{aligned} f(P) &= f(\lambda(P_1 - P_0) + \mu(P_2 - P_0) + P_0) = \\ &= \lambda(f(P_1) - f(P_0)) + \mu(f(P_2) - f(P_0)) + f(P_0); \\ g(P) &= f(\lambda(P_1 - P_0) + \mu(P_2 - P_0) + P_0) = \\ &= \lambda(g(P_1) - g(P_0)) + \mu(g(P_2) - g(P_0)) + g(P_0). \end{aligned}$$

Thus $f(P) = g(P)$ for every point P in the plane. □

For future reference, we single out the information we have about the inverce of a matrix linear function.

COROLLARY 9.1. *Let $f(X) = MX + B$ be a matrix linear function. The f has an inverse if and only if $\Delta(M) \neq 0$. Furthermore, if it has an inverse, that inverse is $f(X) = M^{-1}X - M^{-1}B$.*

As we have seen, matrix linear functions that do not map everything onto a point or line may be visualized as transformations of the plane. Furthermore, their action is completely determined by their action on any three non-collinear points. Hence it is not surprising that a careful choice of those three points yields insight into the transformation. By direct computation we have the following two lemmas.

LEMMA 22. *Consider the matrix linear function*

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

and the non-collinear points $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. *Then* $f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, $f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ *and* $f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

LEMMA 23. *Let* $P_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, $P_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ *and* $P_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ *be any three points. Then*

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} (x_1 - x_0) & (x_2 - x_0) \\ (y_1 - y_0) & (y_2 - y_0) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

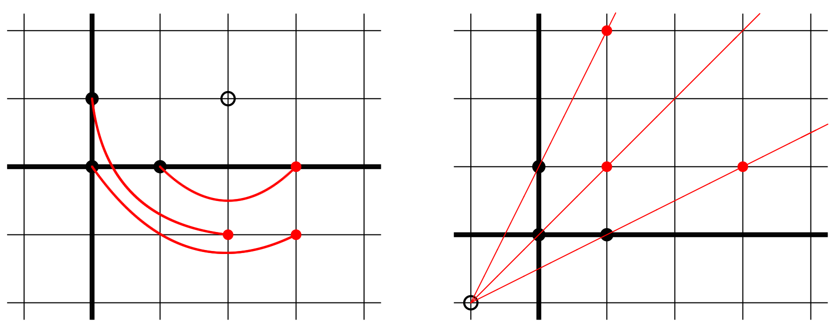
is the unique matrix linear function that maps $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *onto* P_0*,* $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *onto* P_1 *and* $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *onto* P_2*.*

If $P_0 = P_1 = P_2$*, then the image of* f *is this point; if* P_0*,* P_1 *and* P_2 *are not all equal but are collinear, then the image of* f *is the line containing them; if* P_0*,* P_1 *and* P_2 *are non-collinear, then the image of* f *is the entire plane.*

These very simple lemmas are very useful in working with examples and it can be used in either direction: given the formula of a matrix linear function, derive the geometric description or given the geometric description of a matrix linear function, find its formula. For example:

Find the formula for the 90 degree counterclockwise rotation about the point $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. From the picture on the left below, we see that $f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. So $B = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$; the first column of $M = \begin{bmatrix} 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and the second column of $M = \begin{bmatrix} 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. The matrix linear function that rotates the plane 90 degrees counterclockwise about the point $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$



For an example of going in the other direction, consider the matrix linear function

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We compute $f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. In the figure to the right above, we plot these points and their images and observe that this is the dilation with magnification 2 and center $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

Of course, in working out these two exercises, we have assumed that congruences and similarities are always given by a matrix linear function. We will eventually prove this and, for now, adopt it as a working hypothesis as you work through the following exercises.

EXERCISE 6.18. *Give a geometric description for each of the following matrix linear functions.*

(i) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(ii) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(iii) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

(iv) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(v) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(vi) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(vii) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(viii) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(ix) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(x) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The class of matrix linear functions includes many more functions than the similarities and congruences of the plane. For example,

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

maps the entire plane onto the x-axis by projecting each point onto its x coordinate. The matrix linear function

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is one to one and onto; but it distorts distances by different amounts in different directions. Horizontal lines are translated onto themselves by f while vertical lines are mapped onto the 45 degree line intersecting it on the x-axis; these lines are magnified by $\sqrt{2}$.

EXERCISE 6.19. *Give a geometric description for each of the following matrix linear functions.*

(i) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(ii) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(iii) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(iv) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

EXERCISE 6.20. *Find the matrix linear function for each of following geometric transformations.*

(i) *The 180 degree rotation about the point* $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

(ii) *The 90 degree counterclockwise rotation about the point* $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

(iii) *The reflection through the line* $y = x$.

(iv) *The reflection through the line* $y + x = 1$.

(v) *A projection onto the line* $y = x$.

(vi) *The dilation with magnification 2 and center* $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

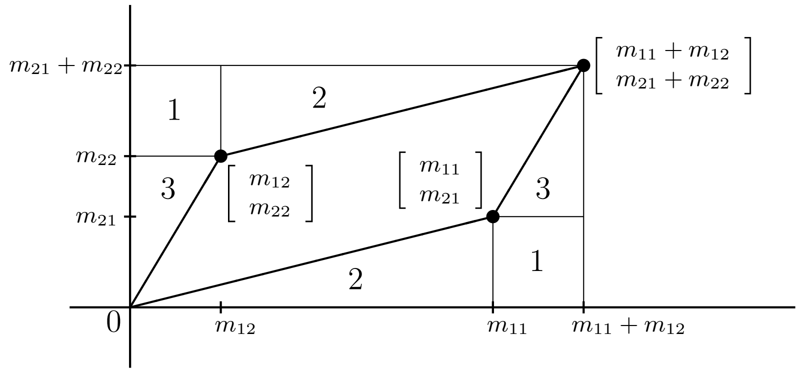
6.4. Affinities and the Determinant

A matrix linear function that maps the plane one-to-one onto itself is called an *affinity* Affinities are distinguished from the matrix linear functions that collapse the plane by having a non-zero determinant. In this section, we will show the determinant ΔM encodes a great deal of information about the matrix linear function $f(X) = MX + B$: it tells us just how f alters area and the orientation of triples of points.

LEMMA 24. *Let* $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ *be any* 2×2 *matrix. Then* $|\Delta(M)|$ *is the area of the quadrilateral with* $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix}$, $\begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix}$ *and* $\begin{bmatrix} m_{11} + m_{12} \\ m_{21} + m_{22} \end{bmatrix}$ *as vertices. Furthermore, if* $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix}$ *and* $\begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix}$ *are noncollinear and oriented counterclockwise,* $\Delta(M)$ *is positive; if they are noncollinear and oriented clockwise,* $\Delta(M)$ *is negative; and, if they are collinear,* $\Delta(M) = 0$.

PROOF. Suppose that the column vectors that make up M are positioned as pictures below. The area of the parallelogram is the area of the rectangle, $(m_{11} + m_{12})(m_{21} + m_{22})$,minus the labeled areas. The two regions labeled 1 contribute $-2m_{12}m_{21}$, the two regions labeled 2 contribute $-m_{11}m_{21}$ and the two regions labeled 3 contribute $-m_{12}m_{22}$. Hence, the area of the parallelogram is:

$$(m_{11} + m_{12})(m_{21} + m_{22}) - 2m_{12}m_{21} - m_{11}m_{21} - m_{12}m_{22} = m_{11}m_{22} - m_{12}m_{21}.$$

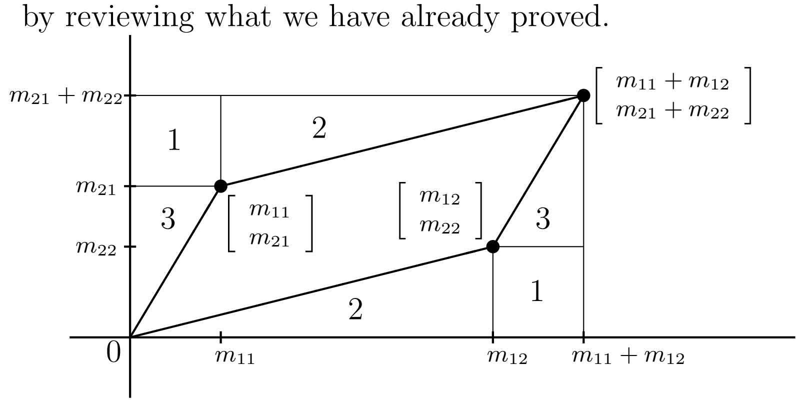


Furthermore, in this case, $m_{11}m_{22} - m_{12}m_{21}$ equals an area and hence is positive, while $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix}$ and $\begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix}$ are oriented counterclockwise. If the the order were clockwise, as pictured below, the positive area would equal the negative of the determinant:

$$(m_{11}+m_{12})(m_{21}+m_{22}) - 2m_{11}m_{22} - m_{11}m_{21} - m_{12}m_{22} = -m_{11}m_{22} + m_{12}m_{21}.$$

Hence, in this case the determinant is negative. Finally, if $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix}$ and $\begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix}$ are collinear, the determinant is 0 and parallelogram collapses and has area 0.

Of course, to give a complete proof, we have to consider all possible configurations of the column vectors $\begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix}$ and $\begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix}$. We can significantly shorten our case by case consideration if we take advantage of some simple geometric arguments. Let's introduce these arguments by reviewing what we have already proved.



Observe first that, in the first quadrant, Z, the zero vector, M_1 the first column of M and M_2, the second column of M, are oriented counterclockwise when M_1 is to the right of M_2, when viewed from the origin. Reflecting the plane through the positive 45 degree line reverses the orientation of these three vectors - indeed the orientation of any three vectors. In terms of the matrix M, reflecting its column vectors through the the positive 45 degree line corresponds to interchanging the first and second rows of M to get M'. So, if Z, M_1 and M_2 are oriented counterclockwise (clockwise), then Z, M_1' and M_2' are oriented clockwise (counterclockwise). Next it is very to check that $\Delta(M') = -\Delta(M)$. Hence, if we complete the proof of our result in the case that Z, M_1 and M_2 are oriented counterclockwise, we may apply this reflection to prove the case for Z, M_1 and M_2 oriented clockwise.

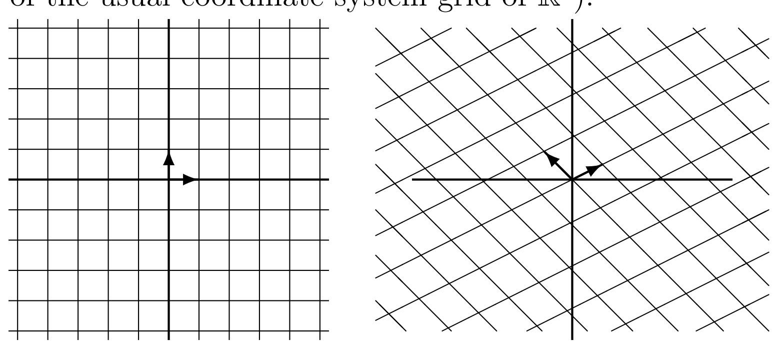
Reflection through the y-axis corresponds to multiplying the each entry in the first row of M by -1 to get M'. Again one easily checks that $\Delta(M') = -\Delta(M)$. We may use this reflection to deduce the result for the case where M_1 and M_2 lie in the second quadrant from the case where M_1 and M_2 lie in the first quadrant. Reflection through the x-axis corresponds to multiplying the each entry in the second row of M by -1 to get M'. Using combinations of these reflections, we can easily take care of all cases in which M_1 and M_2 lie in the same quadrant.

If we can verify our result in the cases where M_1 is in the first quadrant and M_2 is in the second quadrant and where M_1 is in the first quadrant and M_2 is in the third quadrant, then we can complete the proof using combinations of the three reflections introduced above. We leave the proof of these two basic cases as exercises. □

EXERCISE 6.21. *Complete this proof.*

The matrix linear functions that collapse the plane onto a line or point are relatively uninteresting. So we move on to a study of the affinities.

Consider the matrix linear function $f(X) = MX$ where $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ and assume that $|\Delta(M)| \neq 0$. Next consider the image under f of the usual coordinate system grid of $\mathbb{R}^2$):



Note that each parallelogram in the image grid has area $|\Delta(M)|$. If we were to subdivide the initial grid by including the vertical and horizontal lines every tenth of a unit, each parallelogram in the image grid would area $\frac{1}{100}|\Delta(M)|$. Now let $\mathcal{R}$ be a region of the plane that is the union of squares from some subdivision of the initial grid. Then the region $f(\mathcal{R})$ is the union of the corresponding parallelograms from the image grid. We conclude that $f(\mathcal{R})$, has area $|\Delta(M)|\alpha(\mathcal{R})$, where $\alpha(\mathcal{R})$ denoted the area of $\mathcal{R}$. Since the area of any "reasonable" region can be be approximated arbitrarily close by a union of squares from some subdivision of the initial grid, we conclude that f multiplies all areas by $|\Delta(M)|$.

Assume that $|\Delta(M)| > 0$ and let P, Q and R be any noncollinear centers of cells of some subdivision of the initial grid. One easily checks that:

(i) if P, Q and R are orientated clockwise than $f(P)$, $f(Q)$ and $f(R)$ are noncollinear and orientated clockwise.
(ii) if P, Q and R are orientated counterclockwise than $f(P)$, $f(Q)$ and $f(R)$ are noncollinear and orientated counterclockwise.

Similarly, when $|\Delta(M)| < 0$ and P, Q and R are noncollinear centers of cells s of some subdivision of the initial grid. Then

(i) if P, Q and R are orientated clockwise than $f(P)$, $f(Q)$ and $f(R)$ are noncollinear and orientated counterclockwise.
(ii) if P, Q and R are orientated counterclockwise than $f(P)$, $f(Q)$ and $f(R)$ are noncollinear and orientated clockwise.

We have proved:

THEOREM 10. *Consider the affinity* $f(X) = MX + B$ *where* $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ *and* $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. *Then:*

(i) *For any region* $\mathcal{R}$ *in the plane with area* $\alpha(\mathcal{R})$*, the image region,* $f(\mathcal{R})$*, has area* $|\Delta(M)|\alpha(\mathcal{R})$*.*
(ii) *If* $\Delta(M) > 0$ *and* P*,* Q *and* R *are any three noncollinear points then* $f(P)$*,* $f(Q)$ *and* $f(R)$ *are noncollinear and have the same orientation.*
(iii) *If* $\Delta(M) < 0$ *and* P*,* Q *and* R *are any three noncollinear points then* $f(P)$*,* $f(Q)$ *and* $f(R)$ *are noncollinear but have the opposite orientation.*

We say that an affinity $f(X)$ is *direct* if, for every noncollinear triple P, Q and R, the triple $f(P)$, $f(Q)$ and $f(R)$ is noncollinear and has the same (clockwise or counterclockwise) orientation; we say that $f(X)$ is *opposite* if, for every noncollinear triple P, Q and R, the triple $f(P)$, $f(Q)$ and $f(R)$ is noncollinear and has the opposite orientations. We then have:

COROLLARY 10.1. *Let $f(X) = MX + B$ be an affinity. Then f is direct if and only if $\Delta(M) > 0$ and f is opposite if and only if $\Delta(M) < 0$.*

Affinities are the natural analog to the 1-dimensional linear functions, $f(x) = mx + b$. Here m itself plays the role of the determinant. Had we permitted m to equal 0, the linear functions with $m = 0$ would collapse the entire line onto the point b. Reorganizing results that we proved in our study of these functions, we have that for any linear function $f(x) = mx + b$:

(i) For any segment $\mathcal{R}$ in the line with length (1-dimensional area) $\alpha(\mathcal{R})$, the image segment, $f(\mathcal{R})$, has length $|m|\alpha(\mathcal{R})$.
(ii) If $|m| > 0$ and P and Q are any distinct points then $f(P)$ and $f(Q)$ are distinct and are in the same order.
(iii) If $|m| > 0$ and P and Q are any distinct points then $f(P)$ and $f(Q)$ are distinct and are in the reverse order.

While affinities expand or shrink all areas by the same amount, they need not expand or shrink all lengths by the same amount. In an earlier exercise, you considered $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ which doubles some lengths and halves others. We would like to single out from among all affinities those that expand or shrink all lengths by the same amount, namely the similarities. But before we investigate the similarities, we explore a few of the basic properties common to all affinities.

LEMMA 25.

(i) *The composition of affinities is an affinity, in particular:*
- *the composition of two affinities with magnification factors m and n is an affinity with magnification factor mn;*
- *the composition of two direct affinities is a direct affinity;*
- *the composition of two opposite affinities is an opposite affinity;*
- *the composition of a direct affinity and an opposite affinity is an opposite affinity.*

(ii) *For any two 2×2 matrices M and N, $\Delta(MN) = \Delta(M)\Delta(N)$.*

PROOF. Consider the affinities $f(X) = MX + B$, $g(X) = NX + D$ with magnification factors m and n, respectively, and let $h = g \circ f$. Clearly h is a matrix liner function: $h(X) = N(MX + B) + C = NMX + (NB + C)$. If $\mathcal{R}$ is any region, then $area(f(\mathcal{R})) = m \times area(\mathcal{R})$ and $area(h(\mathcal{R})) = n \times m \times area(\mathcal{R})$. So h does not collapse the plane and is, therefore, an affinity. Furthermore, it magnification factor is mn. If both f and g preserve orientation, then h does too. If f reverses orientation and g preserves orientation, then the orientation has been reversed once and h reverses orientation; similarly, if f preserves orientation and g reverses orientation, h reverses orientation. Finally, if both f and g reverse orientation, then the orientation has been reversed twice - back to its original orientation - and h preserves orientation.

Part (ii) follows from Part (i) □

EXERCISE 6.22. *Give a computational proof of Lemma* **??** *(ii)*

EXERCISE 6.23. *In Exercise* **??**, *you gave a geometric description for each of the following matrix functions:*

(i) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(ii) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(iii) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

(iv) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(v) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(vi) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(vii) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(viii) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

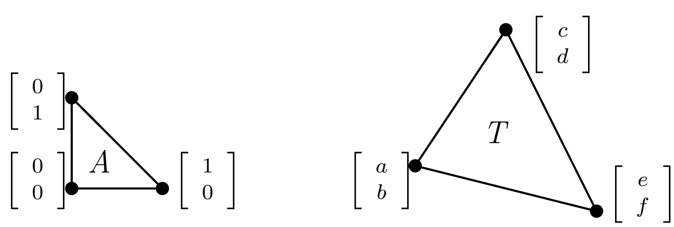
(ix) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(x) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Compute the determinent for each matrix and identify the effect of each on area and orientation. Correlate this with your geometric descriptions.

We may use the results that we have developed for affinities to get a formula for the area of a triangle in $\mathbb{R}^2$ in terms of the coordinates of its vertices.

EXERCISE 6.24. *Consider triangle T with vertices $\begin{bmatrix} a \\ b \end{bmatrix}$, $\begin{bmatrix} c \\ d \end{bmatrix}$ and $\begin{bmatrix} e \\ f \end{bmatrix}$ in clockwise order around the triangle.*



(i) *Show that the direct affinity that that maps* $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *onto* $\begin{bmatrix} a \\ b \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *onto* $\begin{bmatrix} c \\ d \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *onto* $\begin{bmatrix} e \\ f \end{bmatrix}$ *and, hence, triangle* A *onto triangle* T *is:*

$$\begin{bmatrix} e-a & c-a \\ f-b & d-b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}.$$

(ii) *Show that* $\Delta \begin{bmatrix} e-a & c-a \\ f-b & d-b \end{bmatrix} = (af + cb + ed) - (ad + cf + ed)$

(iii) *Conclude that the area of triangle* T *is* $\frac{(af+cb+ed)-(ad+cf+ed)}{2}$.

(iv) *Use this matrix to give a formula for the coordinates of the fourth vertex of a parallelogram in terms of the coordinates of the other three vertices.*

(v) *Use the formula you just derived to show that the sums of coordinates of opposite vertices of a parallelogram are equal.*

(vi) *Use this last observation to show that the diagonals of a parallelogram bisect one another.*

6.5. Fixed Points

Let $f(X) = MX + B$ be any affinity. Suppose that C is a fixed point for $f(X)$. Then we have:

$$C = IC = f(C) = MC + B \quad \text{and} \quad (I - M)C = B.$$

Now, if $(I - M)$ has an inverse, that is if $\Delta(I - M) \neq 0$, then there is a unique choice for C: $C = (I - M)^{-1}B$. We have proved the first part of:

LEMMA 26. *Consider the affinity* $f(X) = MX + B$.

If $\Delta(I - M) \neq 0$, *then* f *has a unique fixed point* $C = (I - M)^{-1}B$ *and can be written in slope-center form:* $f(X) = MX + (I - M)C$.

If $\Delta(I-M) = 0$ *then either* f *has no fixed points, has a line of fixed points or fixes every point in the plane, i.e. is the identity function.*

PROOF. To complete the proof assume that $\Delta(I - M) = 0$ and consider the new matrix linear function $g(X) = (I - M)(X) - B$. One easily checks that the fixed points of f are precisely the points that are mapped onto the origin Z by g. Since $\Delta(I - M) = 0$, the image of g is a line or a point. If Z is not in the image of g, there f has no fixed points and, if g maps the entire plane onto Z, every point is fixed by f. Assume then that the image of g is a line through Z. Then, as you will show in the next exercise, the set of points mapped by g onto Z, that is the fixed points of f, form a line. □

EXERCISE 6.25. *Show that, if the image of the matrix linear function $g(X) = MX + B$ is a line ℓ and P is any point on that line then the set of points $\{X|g(X) = P\}$ is a line.*

As in the real and complex cases, translations, other than the identity, $t(X) = X + B$ ($B \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$) have no fixed points. But there other affinities that have no fixed points and some that have many fixed points. We will investigate both types later.

The key to computing the inverse or center of an affinity, is the ability to compute the inverse of a matrix. A simple formula for the inverse of a 2×2 matrix was given in Lemma **??** (iii):
If $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ and $\Delta(M) \neq 0$, then $M^{-1} = \begin{bmatrix} \frac{m_{22}}{\Delta} & -\frac{m_{21}}{\Delta} \\ -\frac{m_{12}}{\Delta} & \frac{m_{11}}{\Delta} \end{bmatrix}$. Of course any scientific calculator will compute the inverse of a matrix for you.

EXERCISE 6.26. *Compute the inverse of each of the following affinities and compute the centers of those that have a unique fixed point.*

(i) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = I \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

(ii) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

(iii) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

(iv) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

(v) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(vi) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(vii) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

Give a geometric description for each these affinities and their inverses.

The formula for the determinant of a 3×3 matrix is

$$\Delta\left(\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}\right) = \begin{array}{c} m_{11}m_{22}m_{33} - m_{11}m_{23}m_{32} - m_{12}m_{21}m_{33} \\ m_{12}m_{23}m_{31} + m_{13}m_{21}m_{32} - m_{13}m_{22}m_{31}. \end{array}$$

As in the 2-dimensional case, a 3-dimensional matrix linear function is one to one and onto if and only if the determinant of its matrix is nonzero. Also as in the 2-dimensional case, the determinant has geometric significants: its absolute value is the magnification of the volume of all regions and the sign indicates whether orientations are preserved or reversed.

EXERCISE 6.27. *(3D) Give a geometric description for each these affinities and their inverses.*

(i) $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = I \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$.

(ii) $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$.

(iii) $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(iv) $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

(v) $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(vi) $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

6.6. Similarities

Recall that a similarity of n-dimensional space was defined to be a mapping of n-dimensional space onto itself that multiplies all lengths by the same positive constant m, called the magnification. While we have indicated above that similarities are special affinities, that is not immediately obvious. In fact it is not obvious that a similarity is a matrix linear function and so we must prove that. To accomplish this we need to prove a basic lemma about similarities.

LEMMA 27. *If $s(X)$ and $t(X)$ are similarities and P_0, P_1 and P_2 are three non-collinear points so that $s(P_0) = t(P_0)$, $s(P_1) = t(P_1)$ and $s(P_2) = t(P_2)$, then $s = t$.*

PROOF. Since $\frac{|s(P_0)s(P_1)|}{|P_0P_1|} = \frac{|t(P_0)t(P_1)|}{|P_0P_1|}$, s and t have the same magnification, m. Suppose that $s(P_0)$, $s(P_1)$ and $s(P_2)$ were collinear, then $|s(P_i)s(P_j)| = |s(P_i)s(P_k)| + |s(P_k)s(P_j)|$ for some permutation of the indices. But then $|P_i)P_j)| = |P_i)P_k)| + |P_k)P_j)|$ for this same permutation, contradicting the fact that P_0, P_1 and P_2 are non-collinear. Hence $s(P_0)$, $s(P_1)$ and $s(P_2)$ are non-collinear too.

Now let P be any point and we must show that $s(P) = t(P)$. We have $|t(P) - t(P_i)| = m|PP_i| = |s(P) - s(P_i)|$, for each index $i = 0, 1, 2$. For each index $i = 0, 1, 2$, let C_i denote the circle of radius $m|PP_i|$ with center $s(P_i) = t(P_i)$ and note that both $s(P)$ and $t(P)$ must lie on all three circles. If $s(P) \neq t(P)$, these three circles would have to have two distinct points common to all three; but, this can happen only if their centers were collinear and they are not. $\square$

THEOREM 11. *Let $s(X)$ be any similarity. Then s is an affinity: $s(X) = MX + B$, where either*

$$M = m\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \text{ or } M = m\begin{bmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{bmatrix},$$

for some some positive scalar m and some angle θ in the range $0 \leq \theta \leq 2\pi$.

Furthermore, $s(X)$ is direct if and only if $M = m\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$ and $s(X)$ is opposite if and only if $M = m\begin{bmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{bmatrix}$.

PROOF. The first step is to identify the unique matrix linear function that agrees with s on the points $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The second step will be to show that this matrix linear function is a similarity. Then by the previous lemma s will equal this matrix linear function.

Let

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

where $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = s\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$, $\begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} = s\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) - s\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$ and $\begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} = s\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) - s\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$.

We may think of $\begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix}$ as the vector from $s\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$ to $s\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$. Hence its length is the is the distance between $s\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$ and $s\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$. Since $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are a distance one apart, $\left\|\begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix}\right\| = m$, the magnification of s, giving the equation $\sqrt{m_{11}^2 + m_{21}^2} = m$. A similar argument gives the equation $\sqrt{m_{12}^2 + m_{22}^2} = m$. One also easily checks that the distance between $s\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ and $s\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ is $\left\|\begin{bmatrix} m_{11} - m_{12} \\ m_{21} - m_{22} \end{bmatrix}\right\| = m\sqrt{2}$, giving the equation $\sqrt{(m_{11} - m_{12})^2 + (m_{21} - m_{22})^2} = m\sqrt{2}$. Squaring both sides of each of these three equations and dividing through by m gives the following system of three quadratic equations:

$a_{11}^2 + a_{21}^2 = 1$, $a_{12}^2 + a_{22}^2 = 1$ and $(a_{11} - a_{12})^2 + (a_{21} - a_{22})^2 = 2$,
where $a_{ij} = \frac{m_{ij}}{m}$. Expanding the third equation, we have

$a_{11}^2 - 2a_{11}a_{12} + a_{12}^2 + a_{21}^2 - 2a_{21}a_{22} + a_{22}^2 = 2$

and subtracting the first two equations from this gives $a_{11}a_{12} + a_{21}a_{22} = 0$. We conclude that the entries in the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ must satisfy the following system of quadratic equations:

$$a_{11}^2 + a_{21}^2 = 1, \quad a_{12}^2 + a_{22}^2 = 1, \quad a_{11}a_{12} + a_{21}a_{22} = 0.$$

Considering the first equation, we see that $|a_{11}| \leq 1$ and, hence, $a_{11} = cos\theta$ for some angle θ in the range $0 \leq \theta \leq \pi$. It follows that $a_{21} = \pm sin\theta$. However, if we extend the range for θ to $0 \leq \theta \leq 2\pi$ we may select θ so that $a_{11} = cos\theta$ and $a_{21} = sin\theta$. Similarly, we may take $a_{22} = cos\phi$ and $a_{12} = sin\phi$ for some angle $0 \leq \phi \leq 2\pi$.

Now the third equation becomes $cos\theta \sin\phi + sin\theta \cos\phi = 0$ or simply $\sin(\theta+\phi) = 0$. Hence either $\phi = -\theta$ or $\phi = \pi - \theta$ In the first case, we have $a_{12} = -sin\theta$ and $a_{22} = cos\theta$ and, in the second case, we have $a_{12} = sin\theta$ and $a_{22} = -cos\theta$. So the matrix linear function that agrees with s at the three points $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has the form:

$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = m \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ or

$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = m \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

Our next task is to show that both of these matrix linear functions are similarities. Consider the first one and let $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x' \\ y' \end{bmatrix}$ be any two points in the plane. Then:

$$\left| f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right| = \left| m \begin{bmatrix} xcos\theta - ysin\theta \\ xsin\theta + ycos\theta \end{bmatrix} - m \begin{bmatrix} x'cos\theta - y'sin\theta \\ x'sin\theta + y'cos\theta \end{bmatrix}\right| =$$
$$= \left| m \begin{bmatrix} (x-x')cos\theta - (y-y')sin\theta \\ (x-x')sin\theta + (y-y')cos\theta \end{bmatrix}\right| =$$
$$= m\sqrt{(x-x')^2 + (y-y')^2} = m \left| \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \right|.$$

We conclude that this choice for f magnifies all distances by m and is, therefor, a similarity; similarly, the second option yields a similarity. Hence s and f are two similarities that agree on a three non-collinear points and therefore must be equal.

Finally, by direct computation:

$$\Delta \left[m \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}\right] = m^2 > 0 \text{ and } \Delta \left[m \begin{bmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{bmatrix}\right] = -m^2 < 0.$$

□

A similarity with magnification 1 preserves all lengths and is called a *congruence*. We start our classification of congruences and other similarities with a key lemma.

LEMMA 28. *Let $s(X) = MX + B$ be a similarity.*

(i) *If $s(X)$ is not a congruence ($m \neq 1$), then $s(X)$ has a unique fixed point (center) $C = (I - M)^{-1}B$ and s may be rewritten in the form:*

$$s(X) = MX + (I - M)C.$$

(ii) *If $s(X)$ is a direct congruence, then either $s(X)$ is a translation and can be written in the form $t_{[B]}(X) = IX + B$ or s*

has a unique fixed point (center) $C = (I - M)^{-1}B$ *and may be rewritten in the form:*

$$s(X) = MX + (I - M)C.$$

(iii) *If* $s(X)$ *is an opposite congruence, then* $s(X)$ *has no fixed points or an entire line of fixed points.*

PROOF. Assume that $s(X)$ is any direct similarity and consider the matrix $(I - M)$. We have:

$$\Delta(I - M) = \Delta\begin{bmatrix} 1 - m\cos\theta & m\sin\theta \\ -msin\theta & 1 - m\cos\theta \end{bmatrix} = (1 - m\cos\theta)^2 + (m\sin\theta)^2.$$

Since this determinant is the sum of two squares it will be zero if and only if each term is zero, i.e. if $1 - mcos\theta = 0$ and $msin\theta = 0$. One easily sees that the only solution to this system of equations is $\theta = 0$ and $m = 1$. But, these two conditions hold only when M=I, i.e. when $s(X) = IX + B$ is a translation. If $s(X)$ is any direct similarity other than a translation, then: $\Delta(I - M) \neq 0$, $I - M$ is invertible and $s(X) = X$ has a unique solution. So $C = (I - M)^{-1}B$ is the unique fixed point for $s(X)$. and $s(X) = MX + (I - M)C$.

Next assume that $s(X)$ is an opposite similarity. Then

$$\Delta(I - M) = \Delta\begin{bmatrix} 1 - m\cos\theta & -m\sin\theta \\ -m\sin\theta & 1 + m\cos\theta \end{bmatrix} = 1 - (m\cos\theta)^2 - (m\sin\theta)^2 = 1 - m^2.$$

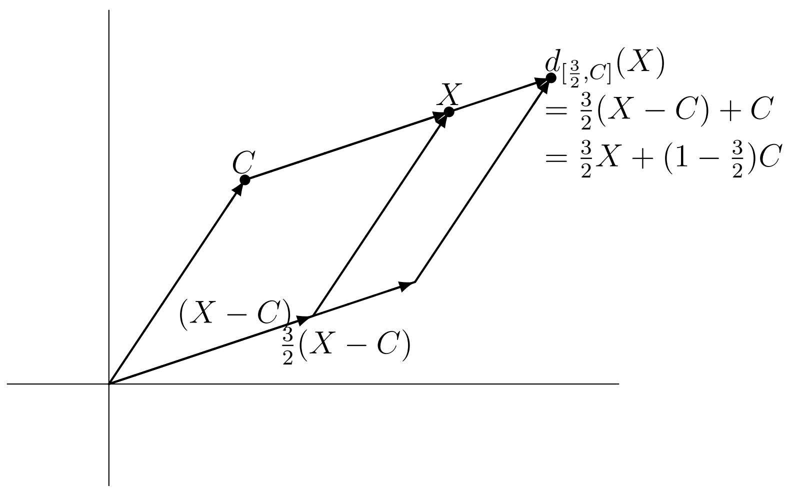
Hence, in this case, $s(X)$ will have a unique fixed point if and only if $m \neq 1$, i.e. if and only if $s(X)$ is not a congruence. If s a congruence, we have by Lemma **??** that f either f has no fixed points, a line of fixed points or is the identity - this last case being impossible since f is opposite. □

EXERCISE 6.28. *Let* $s(X)$ *be a similarity that fixes the two distinct points* P *and* Q*. Using the geometric properties of similarities show that*

(i) $s(X)$ *is a congruence.*
(ii) $s(X)$ *fixes all of the points on the line through* P *and* Q*.*
(iii) *If* $s(X)$ *is direct then it is the identity.*
(iv) *If* $s(X)$ *is opposite it is the reflection about the line through* P *and* Q*.*

The simplest of all of the similarities are the dilations: $d(X) = D_mX + B$, where D_m is the diagonal matrix $D_m = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} = mI$, where $m > 0$ and $m \neq 1$. (If we permitted $m = 1$, $d(X)$ would be a translation). The simplest of all dilations are the dilations about the origin: $d(X) = D_mX$. These are very easy to understand: each point is moved toward ($m < 1$) or away from ($m > 1$) the origin. the best way to think of multiplying X by D_m is to think of X as a vector then $D_m X$ is simply the scalar multiple mX. Geometrically, the general dilation $d(X) = D_m X + B$ stretches ($m > 1$) or shrinks ($m < 1$) all distances to a center C by the factor m. This center or fixed point of $d(X)$ is easily computed to be $C = (I - D_m)^{-1}B = D_{(\frac{1}{1-m})}B$. As we just noted, we may interpret multiplication by the matrix D_m as simply multiplication by the scalar m. So $d(X) = mX + B$ or $d(X) = mX + (1-m)C$ where $C = \frac{1}{1-m}B$ is the center of the dilation. We will adopt the notation $d_{[m,C]}(X)$ for the dilation with magnification m and center C.

Geometrically, the dilation $d_{[m,C]}(X)$ stretches ($m > 1$) or shrinks ($m < 1$) all distances to the center C by the factor m:



EXERCISE 6.29. *Prove that the inverse of the dilation* $d_{[m,C]}$ *is* $d_{[\frac{1}{m},C]}$.

LEMMA 29. *Let* $s(X) = MX + (I - M)C$ *be the similarity with magnification* $m \neq 1$ *and center* C. *Then there exist a unique congruence with* C *as a fixed point,* $c(X) = AX + (I - A)C$, *so that* $s(X) = c(d_{[m,C]}(X)) = d_{[m,C]}(c(X))$.

PROOF. Consider the composition $d_{[\frac{1}{m},C]}(s(X))$. All distances are first multiplied by m and then by $\frac{1}{m}$ in this composition. The net effect is to preserve all distances so $d_{[\frac{1}{m},C]}(s(X))$ is a congruence. Furthermore, both $d_{[\frac{1}{m},C]}$ and s leave C fixed so their composition leaves C fixed. We conclude that $d_{[\frac{1}{m},C]}(s(X)) = AX + (I - A)C$, where $A = \begin{bmatrix} cos\theta & \mp sin\theta \\ sin\theta & \pm cos\theta \end{bmatrix}$. Let $c(X) = AX + (I - A)C$ and note that $d_{[m,C]}(c(X)) = d_{[m,C]}(d_{[\frac{1}{m},C]}(s(X))) = s(X)$.

We leave as an exercise for the reader the verification of commutativity: $c(d_{[m,C]}(X)) = d_{[m,C]}(c(X))$. □

Thus, once we have a classification of the congruences we will have a complete classification of the similarities.

6.7. Congruences

Let $f(X)$ be a direct congruence. As we have seen, either $f(X)$ is a translation and has the form $f(X) = X + B$ or $f(X)$ has a fixed point and has the form $f(X) = AX + (I - A)C$ where C is the fixed point and $A = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$ with $0 < \theta \leq 2\pi$. We understand just how a translation acts on the plane and we now wish to understand just how these other direct isometries act on the plane. And to do this we must be able to measure angles.

We define the *inner product* of two vectors $V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ by:

$$V \cdot W = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = v_1 w_1 + v_2 w_w.$$

The *length* of the vector V can be computed using the inner product:

$$|V| = \sqrt{V \cdot V} = \sqrt{v_1^2 + v_2^2}.$$

The *distance* between two points V and W may then computed in terms of the inner product as the length of the difference vector:

$$|VW| = |V - W| = \sqrt{(v_1 - w_1)^2 + (v_2 - w_2)^2}.$$

As with all of the operations that we have introduced, we must clearly understand the rules for manipulating inner products algebraically and, as usual, verifying these rules is straight forward and left as an exercise.

Lemma 30. *Let U, V and W be vectors and s a real number. Then:*

(i) *(Commutativity) $U \cdot V = V \cdot U$;*
(ii) *(Associativity) $(sU) \cdot V = s(U \cdot V)$;*
(iii) *(Distributivity) $(U + V) \cdot W = (U \cdot W) + (V \cdot W)$.*

Exercise 6.30. *Prove Lemma* **??**.

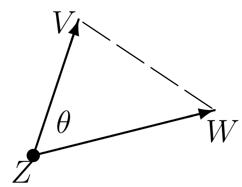
Our next task is to relate the inner product to the measure of the angle between the two vectors. The formula that we produce is just the Law of cosines in disguise.

Lemma 31. *The cosine of the angle θ between vectors V and W, $\angle VZW$ is given by:*

$$cos(\theta) = \frac{V \cdot W}{|V||W|};$$

in particular, $V \perp W$ if and only if $V \cdot W = 0$.

Proof. Consider the triangle VZW.



By the law of cosines: $|V - W|^2 = |V|^2 + |W|^2 - 2|V||W| \cos\theta$. Expanding the left hand side, we have:

$$\begin{aligned} |V - W|^2 &= (V - W) \cdot (V - W) \\ &= V \cdot V - 2V \cdot W + W \cdot W \\ &= |V|^2 + |W|^2 - 2V \cdot W. \end{aligned}$$

Substituting this for the left hand side and canceling, gives

$-2(V \cdot W) = -2|V||W| \cos\theta$. □

Lemma 32. *Let $f(X) = AX + (I - A)C$ be the direct congruence with fixed point C where $A = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$ with $0 < \theta \leq 2\pi$. Then $f(X)$ is the counterclockwise rotation about C through an angle of measure θ.*

Proof. Let P be any point in the plane. Since $f(X)$ preserves distances and since C is mapped onto itself, the vectors $P - C$ and $f(P) - C$ must also must have the same length. We wish to show that the measure of angle $\angle PCf(P) = \theta$. As we have seen,

$$\angle PCf(P) = \frac{(P - C) \cdot (f(P) - C)}{|P - C||f(P) - C|} = \frac{(P - C) \cdot (f(P) - C)}{|P - C|^2}.$$

Now let $(P - C) = \begin{bmatrix} x \\ y \end{bmatrix}$. Then:

$P = \begin{bmatrix} x \\ y \end{bmatrix} + C$,

$f(P) = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + C$ and

$f(P) - C = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xcos\theta - ysin\theta \\ xsin\theta + ycos\theta \end{bmatrix}$

Thus,

$(P-C)\cdot(f(P)-C) = \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} xcos\theta - ysin\theta \\ xsin\theta + ycos\theta \end{bmatrix} = (x^2+y^2)cos\theta = |P-C|^2 cos\theta.$
and
$\angle PCf(P) = \frac{(P-C)\cdot(f(P)-C)}{|P-C|^2} = \frac{|P-C|^2 cos\theta}{|P-C|^2} = cos\theta$, as we were to show. □

We now adopt the notation $r_{[C,\theta]}$ for the rotation by an angle of measure θ about the point C. Combining the previous lemma with Lemma **??** gives:

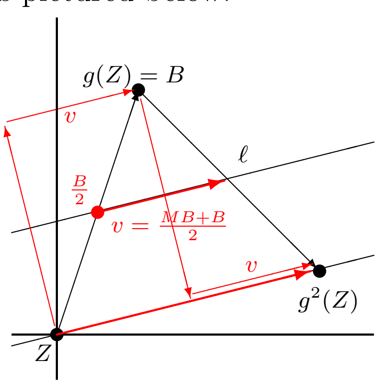
LEMMA 33. *If $s(X)$ is a direct similarity with magnification $m \neq 1$ and center C then $s(X) = d_{[m,C]}(r_{[C,\theta]}) = r_{[C,\theta]}(d_{[m,C]})$.*

EXERCISE 6.31. *Let $s(X) = d_{[m,C]}(r_{[C,\theta]})$ be a direct similarity with $m \neq 1$.*

(i) *Describe s^{-1}.*
(ii) *Give the simple matrix equation for s.*

As we noted above, opposite congruences do not have a unique fixed point. But as we will soon see, they do have an *axis*, a line of fixed points or a fixed line (a line mapped onto itself). Indeed, an opposite congruences is either a reflection or a glide reflection.

For the moment we will assume that this is true and use that assumption to get some insight into how we should go about proving that it is true. Suppose that $g(X) = MX + B$, where $M = \begin{bmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{bmatrix}$, is a glide reflection with axis ℓ and, for now assume that does not pass through the origin, as pictured below.



Under these assumptions, Z (the origin) is reflected across ℓ and then translated by a direction vector for ℓ. Since $g(Z) = MZ + B = B$, $\frac{B}{2}$, the midpoint of the segment ZB, must be on ℓ. To find the translation vector we observe that g^2 is a translation, indeed it is twice the translation part of g:

$$g^2(X) = M(MX + B) + B = M^2X + MB + B = X + (MB + B);$$

(You should check that M^2 is the identity). If $\frac{MB+B}{2} = Z$, g is actually a reflection. We skip this special case for now and assume that $\frac{MB+B}{2} \neq Z$. Under this additional assumption, $\lambda\frac{MB+B}{2} + \frac{B}{2}$ is a parametric equation for ℓ.

We assume that $g(X) = MX + B$, where $M = \begin{bmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{bmatrix}$.

To see that our geometric reasoning actually corresponds to the action of g on the line ℓ, let $P = \lambda_0\frac{MB+B}{2} + \frac{B}{2}$ be a point on ℓ and compute $g(P)$:

$$g(P) = M(\lambda_0\frac{MB+B}{2}+\frac{B}{2})+B = \lambda_0\frac{B+MB}{2}+\frac{MB}{2}+B = \left(\lambda_0\frac{B+MB}{2}+\frac{B}{2}\right)+\frac{MB+B}{2}.$$

Indeed, g translates the line ℓ onto itself by the vector $\frac{MB+B}{2}$.

Now we must check that for points not on ℓ, g actually reflects points through ℓ and then translates them by $\frac{MB+B}{2}$. To avoid loosing the thread of our argument, we make the following observations and leave the proofs to later:

- the points $P_0 = \frac{B}{2}$, $P_1 = \frac{MB}{2} + B$ and $P_2 = \frac{MB}{2}$ are noncolinear and
- the vectors $\frac{MB+B}{2}$ and $\frac{MB-B}{2}$ are perpendicular.

By the first observation, we may coordinatize the plane:

$$P(\lambda,\mu) = P_0 + \lambda(P_1 - P_0) + \mu(P_1 - P_0) = \frac{B}{2} + \lambda\frac{MB+B}{2} + \mu\frac{MB-B}{2}$$

Now compute the action of g on an arbitrary point:

$G\left(P(\lambda,\mu)\right) = M\left(\frac{B}{2} + \lambda\frac{MB+B}{2} + \mu\frac{MB-B}{2}\right) + B$

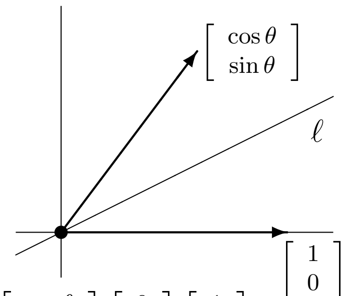
We leave it to the reader to verify that $\frac{MB-B}{2}$ is perpendicular to $\frac{MB+B}{2}$. Assuming this to be true and assuming that $\frac{MB+B}{2}$ is not thezero vector

The best way to understand these isometries is to consider the simplest of these first: $f(X) = AX$, where $A = \begin{bmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{bmatrix}$.

We can argue geometrically as follows:

- $f(X) = AX$ fixes the origin and, therefore must fix many points.
- Hence $f(X)$ fixes a line ℓ.
- $\begin{bmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix}$.
- Let P be a point on ℓ; then the segment joining P and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is mapped onto the segment joining P and $\begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix}$.

- Hence ℓ is the set of points equidistant from $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix}$.
- In other words ℓ is the bisector of the angle $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.



- Since angle $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has measure θ, ℓ makes an angle of $\frac{\theta}{2}$ with the positive x-axis.
- Now is a good time to remember the half angle formulas:
  $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$,
  (+, when $0 \le \theta \le \pi$, and (-, when $\pi \le \theta \le 2\pi$);
  $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$, $0 \le \theta \le 2\pi$.
  The best way to develop these formulas is in the context of a general development of all the trigonometric formulas. However, for the sake of efficency, a direct derivation is outlined in the an exercise below.
- The vector $D = \begin{bmatrix} \sqrt{\frac{1+\cos\theta}{2}} \\ \sqrt{\frac{1-\cos\theta}{2}} \end{bmatrix}$ is a unit direction-vector for ℓ.
- Simpler direction-vectors for ℓ can be computed by multyplying D by $\sqrt{2(1+\cos\theta)}$ to get $D = \begin{bmatrix} 1+cos\theta \\ sin\theta \end{bmatrix}$ or multyplying D by $\sqrt{2(1-\cos\theta)}$ to get $D = \begin{bmatrix} \sin\theta \\ 1-\cos\theta \end{bmatrix}$.

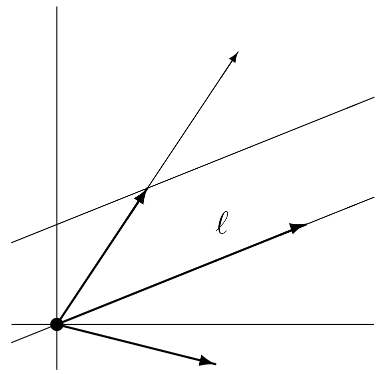
EXERCISE 6.32. *Verify these last computations for the direction-vector D.*

EXERCISE 6.33. *Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the unit vector that bisects the angle between the vectors $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.*

(i) *Compute the square of the distance between X and $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and set it equal to the square of the distance between X and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to get: $(1-\cos\theta)x_1 + (\sin\theta)x_2 = 0$.*
(ii) *solve the system*
$x_1(1-\cos\theta) + x_2\sin\theta = 0$

$x_1^2 + x_2^2 = 1$
to get $x_1^2 = \frac{1+\cos\theta}{2}$ *and* $x_2^2 = \frac{1-\cos\theta}{2}$.

Before we give algebraic proofs of the results of these geometric argument, we continue in this geometric vain and consider the general opposite congruence $g(X) = AX + B$



LEMMA 34. *Let* $g(X) = AX + B$ *be an opposite congruence with* $A = \begin{bmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{bmatrix}$.

(i) *If* $AB + B = 0$*, then* $g(X)$ *is a reflection about the axis with point - direction-vector equation* $\frac{1}{2}B + \lambda D$ *where* $D = \begin{bmatrix} 1 + cos\theta \\ sin\theta \end{bmatrix}$.

(ii) *If* $AB + B \neq 0$*, then* $g(X)$ *is a glide-reflection about the axis with point - direction-vector equation* $\frac{1}{2}B + \lambda D$ *where* $D = \begin{bmatrix} 1 + cos\theta \\ sin\theta \end{bmatrix}$ *and with translation* $\frac{AB+B}{2}$.

PROOF. Throughout this proof we assume that $\sin\theta \neq 0$ and leave that special case as an exercise for the reader. We start the proof by proving that $\frac{AB+B}{2} = \eta D$, for some real number η. A key step in this proof is the observation:

$$\frac{1 - cos\theta}{sin\theta} = \frac{(1 - cos\theta)(1 + cos\theta)}{sin\theta(1 + cos\theta)} = \frac{1 - cos^2\theta}{sin\theta(1 + cos\theta)} = \frac{sin^2\theta}{sin\theta(1 + cos\theta)} = \frac{sin\theta}{1 + cos\theta}.$$

Thus:

$$\begin{aligned} \begin{bmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} &= \begin{bmatrix} (1 + cos\theta)b_1 + (sin\theta)b_2 \\ (sin\theta)b_1 + (1 - cos\theta)b_2 \end{bmatrix} \\ &= \begin{bmatrix} (1 + cos\theta)(b_1 + \frac{sin\theta}{1-cos\theta}b_2) \\ (sin\theta)(b_1 + \frac{1-cos\theta}{sin\theta}b_2) \end{bmatrix} \\ &= (b_1 + \frac{1-cos\theta}{sin\theta}b_2)\begin{bmatrix} 1 + cos\theta \\ sin\theta \end{bmatrix} \end{aligned}$$

Taking $\eta = \frac{1}{2} \times (b_1 + \frac{1-cos\theta}{sin\theta}b_2)$, we have the equality we are seeking.

Now let P be a point on the proposed axis, specifically, let $P = \frac{1}{2}B + \lambda D$, for some real number λ. Then:

$$\begin{aligned} g(P) &= A(\tfrac{1}{2}B + \lambda D) + B \\ &= \tfrac{1}{2}AB + \lambda AD + B \\ &= \tfrac{1}{2}B + \lambda D + \tfrac{AB+B}{2} \\ &= P + (\lambda + \eta)D \end{aligned}$$

Thus, the line $\frac{1}{2}B + \lambda D$ is mapped into itself by $g(X)$ and $g(X)$ translates each point on this line along this line by the vector $\frac{AB+B}{2} = \eta D$.

Assume first that $AB+B = 0$. Since $g(X)$ is an opposite congruence which fixes each point on the line $\frac{1}{2}B + \lambda D$, it is the reflection about that line.

Now suppose that $AB + B \neq 0$. Let $B' = B - \eta D$ and consider the transformation $g'(X) = AX + B'$. First, we note that $g'(X)$ is a reflection:

$$\begin{aligned} AB' + B' &= A(B - \eta D) + B - \eta D \\ &= AB - \eta AD + B - \eta D \\ &= AB + B - 2\eta D = 0 \end{aligned}$$

Finally, we observe that $g(X) = t \circ g'(X)$, where $t(X) = X + \eta D$ is a translation along the axis of the reflection $g'(X)$. Thus $g(X)$ is the glide reflection described in part (ii) above. □

EXERCISE 6.34. *Prove this lemma for the special case* $\sin\theta = 0$.

EXERCISE 6.35. *Let* ℓ *denote the line that has point-direction-vector form* $C + \lambda D$ *and let* η *be any real number. Let* g *be the glide reflection (or reflection when* $\eta = 0$*) consisting of the reflection through* ℓ *followed by the translation* $t_{[\eta D]}$. *Show that you can recover the standard form* $g(X) = AX + B$ *from these parameters. We do this in several steps.*

(i) *Show that the (non-zero) direction-vector* D *can be scaled to have the form* $\lambda D = \begin{bmatrix} 1+\cos\theta \\ \sin\theta \end{bmatrix}$. *Specifically, show that you can solve this system for* λ *and* θ. *Since* C *and* λD *also determine* ℓ, *we may replace* D *by* λD *and simply assume that* $D = \begin{bmatrix} 1+\cos\theta \\ \sin\theta \end{bmatrix}$. *Then we can recover* $A = \begin{bmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{bmatrix}$

(ii) *Next we recover* B *by solving the following system of vector equations*

$$AB + B = 2\eta D \text{ and } B = 2(C + \lambda D) \text{ for } \lambda \text{ and } B.$$

We let $g_{[C,D,\eta]}(X)$ denote the glide reflection with the line with point and direction-vector C and D as axis and the vector ηD as translation. Of course, $g_{[C,D,0]}(X)$ is a reflection.

We may now gather together all of these lemmas to get a complete description of all similarities of the plane.

THEOREM 12.
The direct congruences of the plane are:

(i) *translations,* $t_{[B]}(X)$;
(ii) *rotations,* $r_{[C,\theta]}$.

The opposite congruences of the plane are the reflections and glide reflections: $g_{[C,D,\eta]}(X)$.

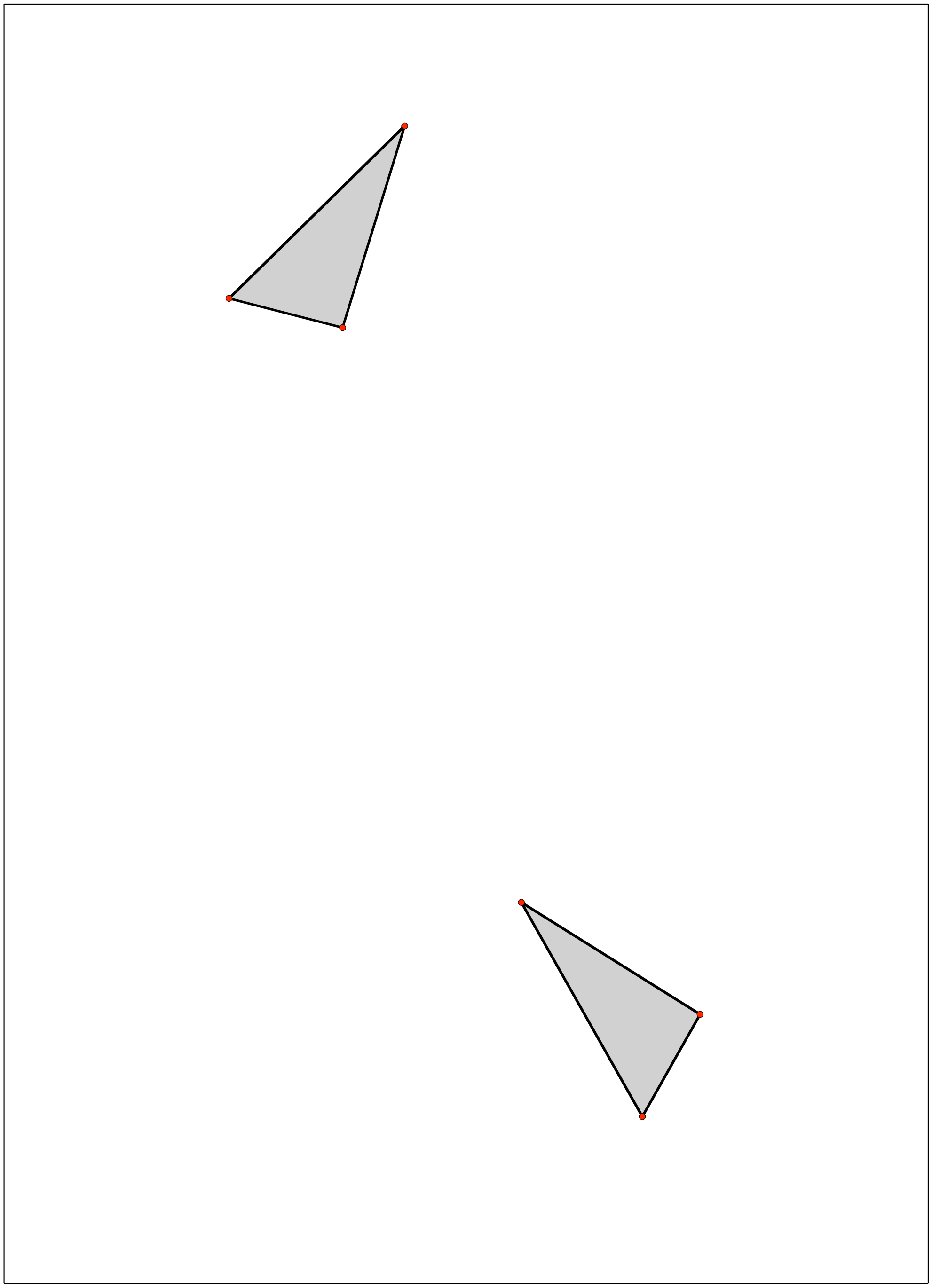
The direct similarities of the plane with magnification different from 1 all have the form $d_{[C,m]}(r_{[C,\theta]}) = r_{[C,\theta]}(d_{[C,m]})$.

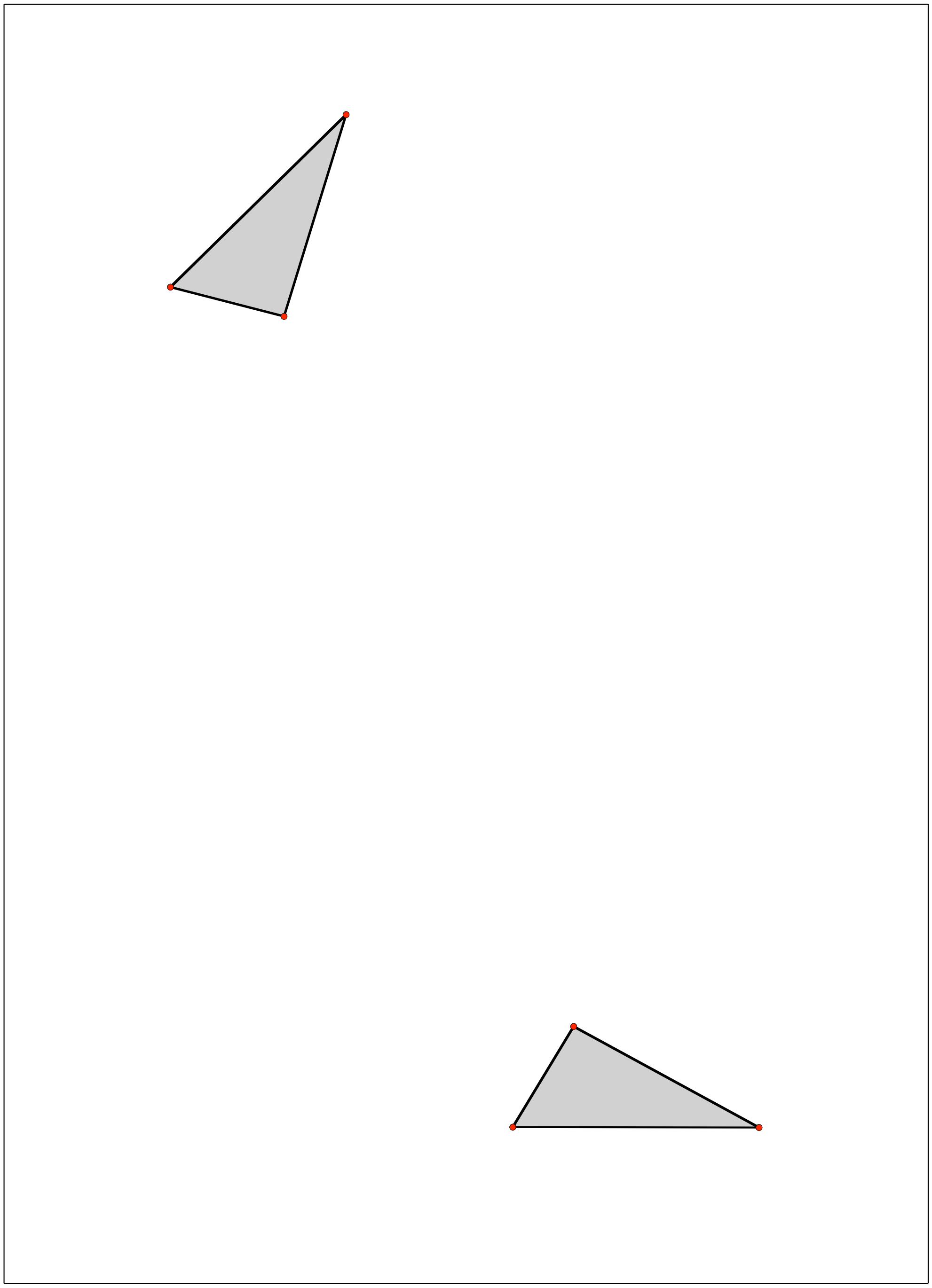
The opposite similarities of the plane with magnification different from 1 all have the form $d_{[C,m]}(g_{[C,D,0]}) = g_{[C,D,0]}(d_{[C,m]})$.

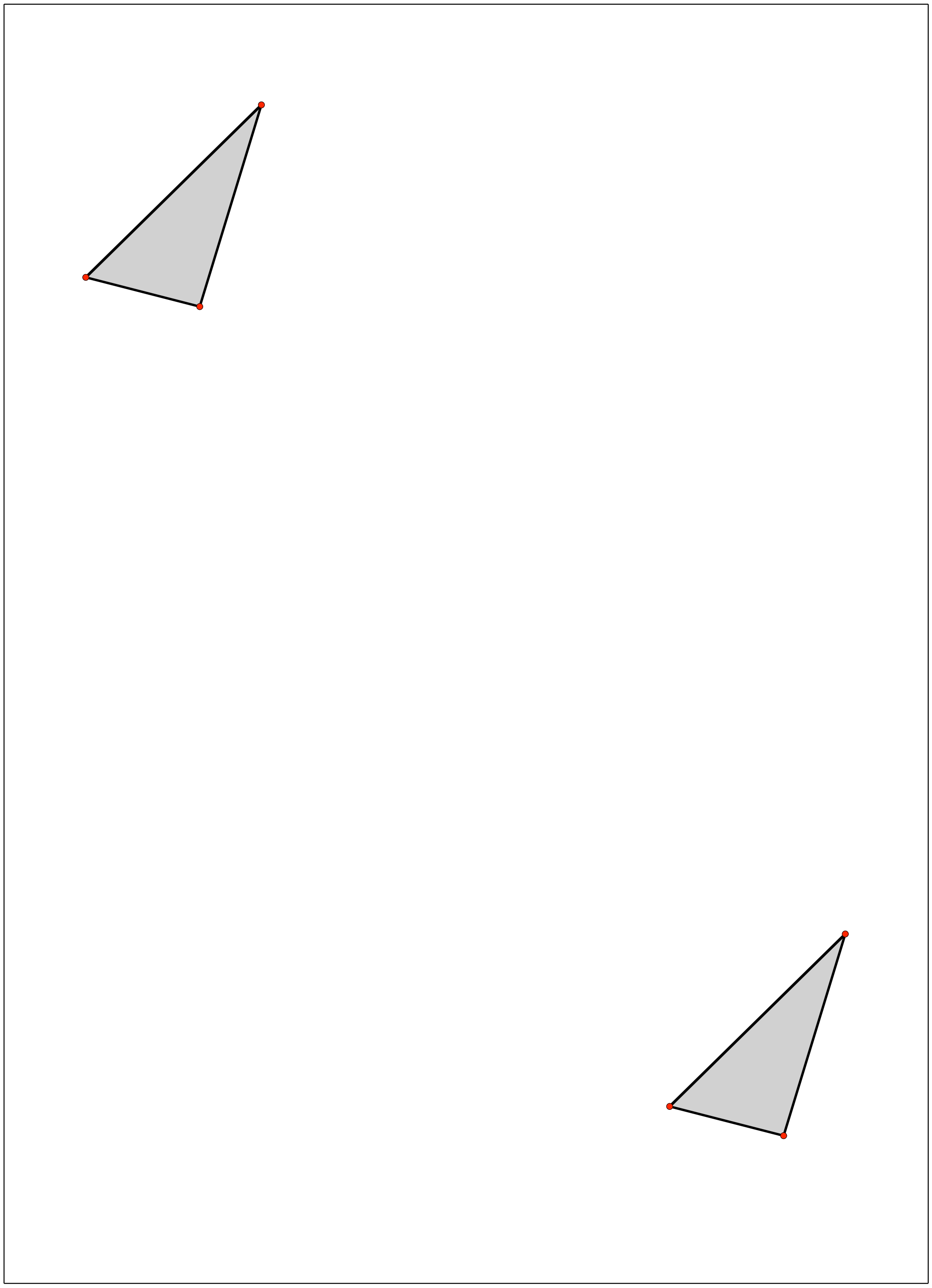
It is clear that a congruence will map a triangle onto a congruent triangle and a similarity will map a triangle onto a similar triangle. It seems reasonable that the converse is true: if triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent, then there exists a congruence c that maps $\triangle ABC$ onto $\triangle A'B'C'$ and, if triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar, then there exists a similarity s that maps $\triangle ABC$ onto $\triangle A'B'C'$. Actually, this fact implicit in our development of congruences and similarities - but it is easy to make it explicit. Assume that $\triangle ABC$ and $\triangle A'B'C'$ are similar and let t be the translation that maps A onto A'. Next let r be the rotation with center A' that maps the segment $A't(B)$ onto a segment containing or contained in the segment $A'B'$. Since $\angle C'A'B' = \angle CAB$, either the segment $A'rt(C)$ is containing or contained in the segment $A'C'$ or $A'grt(C)$ is containing or contained in the segment $A'C'$, where g is the reflection through the line containing the segment $A'B'$. Finally, since the triangles are similar the ratios $m = \frac{|A'B'|}{AB}$ equals the ratio $\frac{|A'C'|}{AC}$ (m=1 in the case the triangles are congruent). So, if $\triangle ABC$ and $\triangle A'B'C'$ are congruent, the congruence grt maps $\triangle ABC$ onto $\triangle A'B'C'$ and, if $\triangle ABC$ and $\triangle A'B'C'$ are similar, the congruence $d_{[m,A']}grt$ maps $\triangle ABC$ onto $\triangle A'B'C'$. In view of Lemma **??**, this similarity or congruence mapping $\triangle ABC$ onto $\triangle A'B'C'$, it is unique.

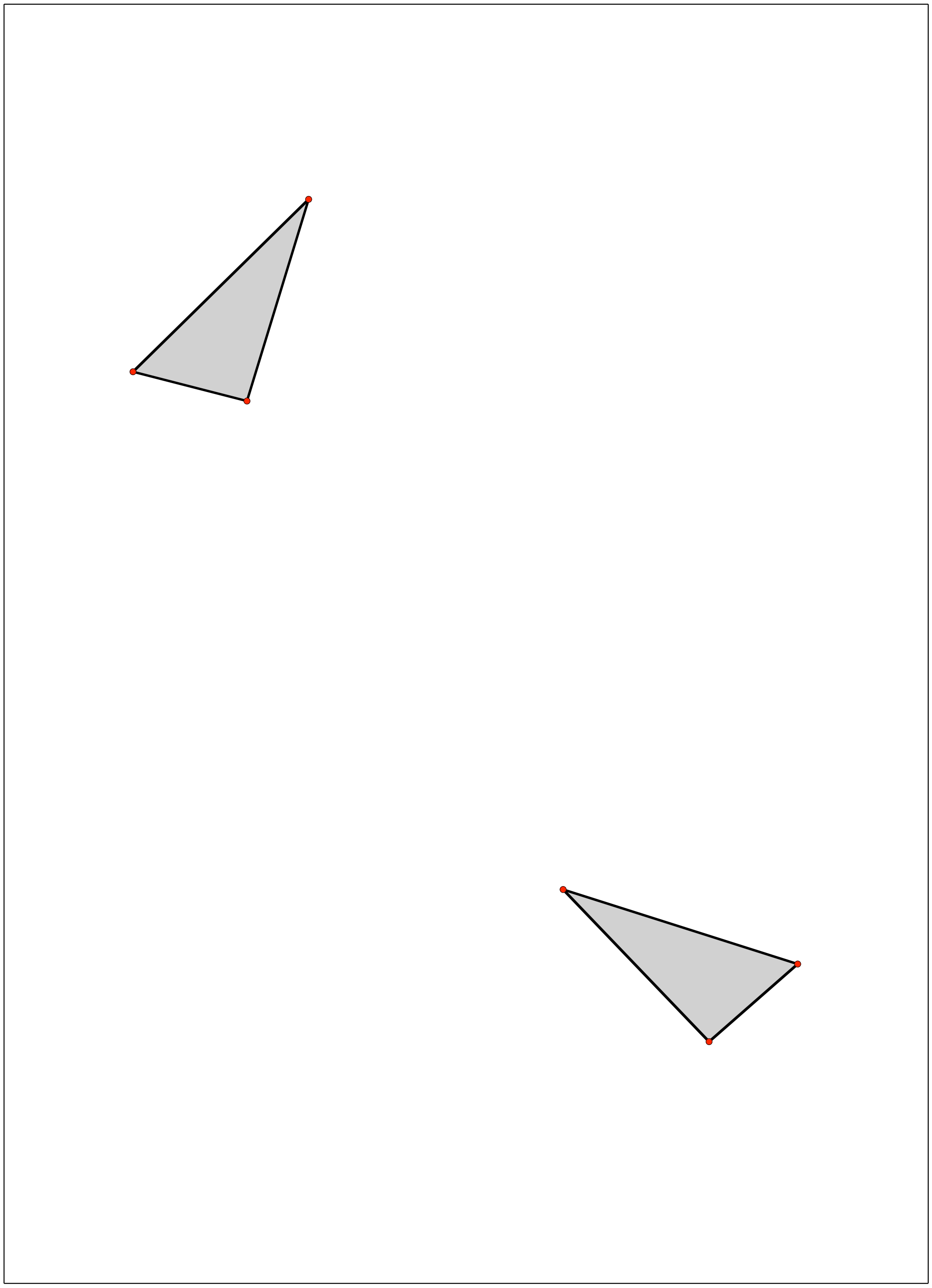
Given two similar or congruent triangles, finding the similarity or congruence that maps one onto the other is not so easy. The above argument gives this transformation as a composition of two, three or four congruences. We could construct these and try to simplify the composition. But this too can be a difficult problem. We start this final exploration in this chapter by listing several problems involving congruences for the reader to try. We then give a few exercises that develop techniques for solving these problems, encouraging the reader to redo the problems using these techniques. We then repeat the sequence for similarities that are not congruences.

EXERCISE 6.36. *In each of the following four cases identify the congruence that maps the topmost triangle onto the other triangle. If it is a translation, give the translation vector; if it is a rotation, identify the center and give the angle of rotation; if it is a reflection or glide reflection, give the axis of reflection and the vector of translation in the case of a glide reflection.*









EXERCISE 6.37. *Prove the following construction lemmas.*

(i) *If* r *is a rotation and* X *is any point then the center of rotation lies on the perpendicular bisector of the segment joining* X *and* $r(X)$.

(ii) *If* g *is a reflection or glide reflection and* X *is any point then the the axis of reflection pass through the midpoint of the segment joining* X *and* $r(X)$.

EXERCISE 6.38. *Consider the coordinate plane and the following congruences:* r_1 *the 90° (counterclockwise) rotation about the point (2,1);* r_1 *the 180° rotation about the point (1,-1);* t *the translation by the vector (1,1);* g_1 *the reflection through the line* $y = 1$; g_2 *the reflection through the line* $x + y = 1$.

Compute the composition of each ordered pair of these congruences.

Turning to the problem of identifying a similarity, we prove the following useful lemma:

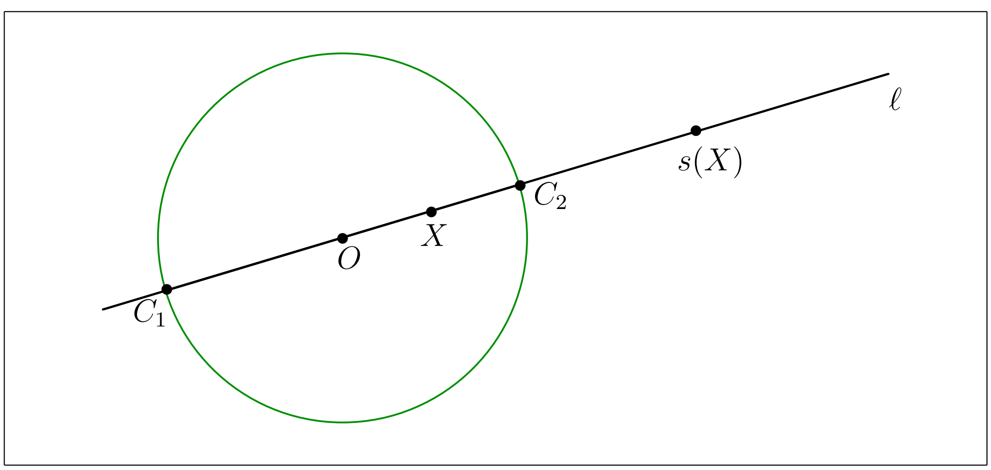
LEMMA 35. *Let* s *be a similarity with magnification* $m \neq 1$ *and let the point* X *be given. Then the center of* s *lies on the circle constructed as follows (illustrated for* $m = 2$ *below):*

(i) *Construct the point* C_1 *on the line* ℓ *through* X *and* $s(X)$ *and outside the interval* $Xs(X)$ *such that* $|C_1 s(X)| = m|C_1 X|$.

(ii) *Construct the point* C_2 *on the line* ℓ *through* X *and* $s(X)$ *and inside the interval* $Xs(X)$ *such that* $|C_2 s(X)| = m|C_2 X|$.

(iii) *Construct* O, *the midpoint of the segment* C_1C_2.

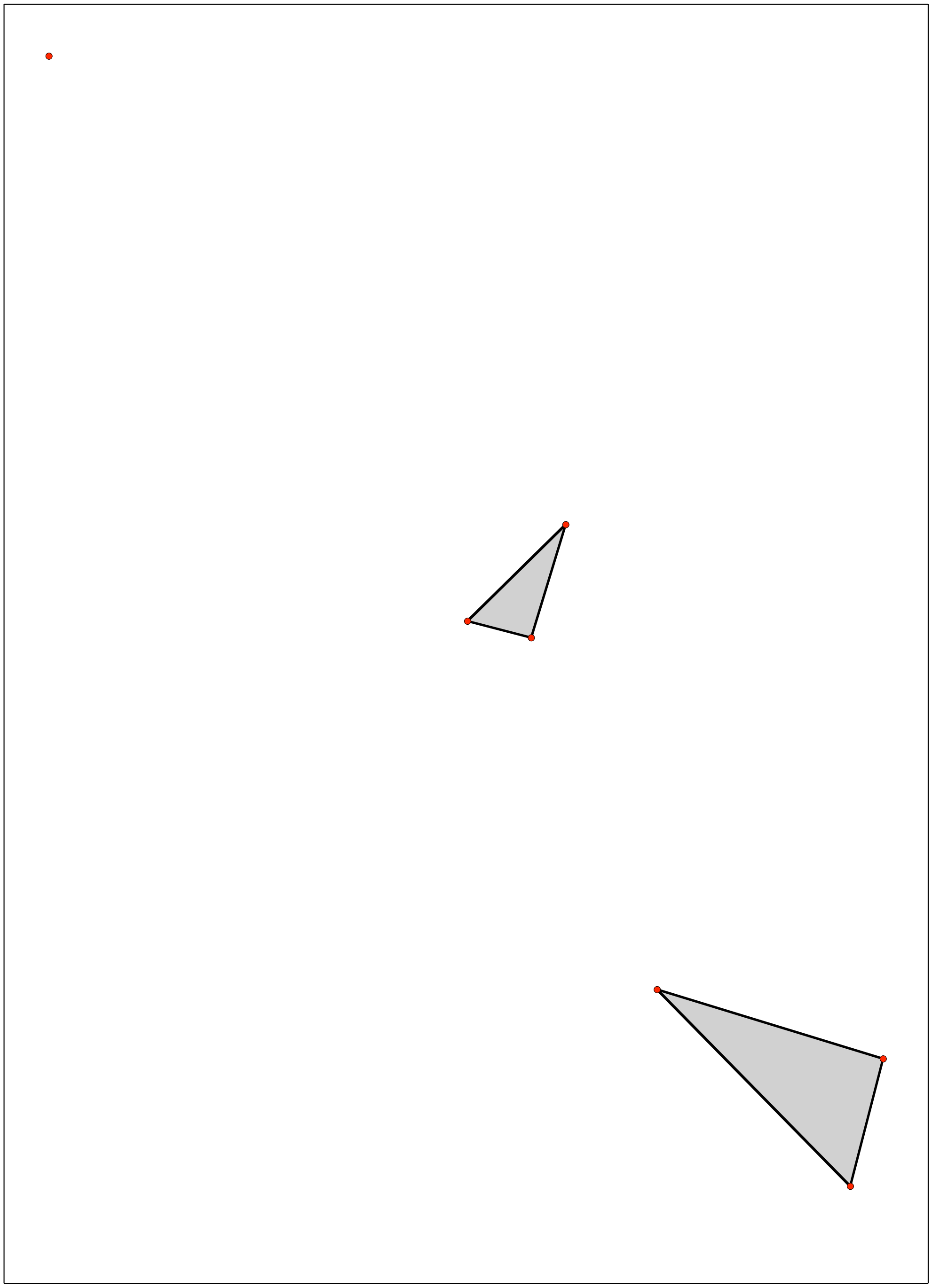
(iv) *Construct the circle with center* O *and passing through* C_1 *and* C_2.

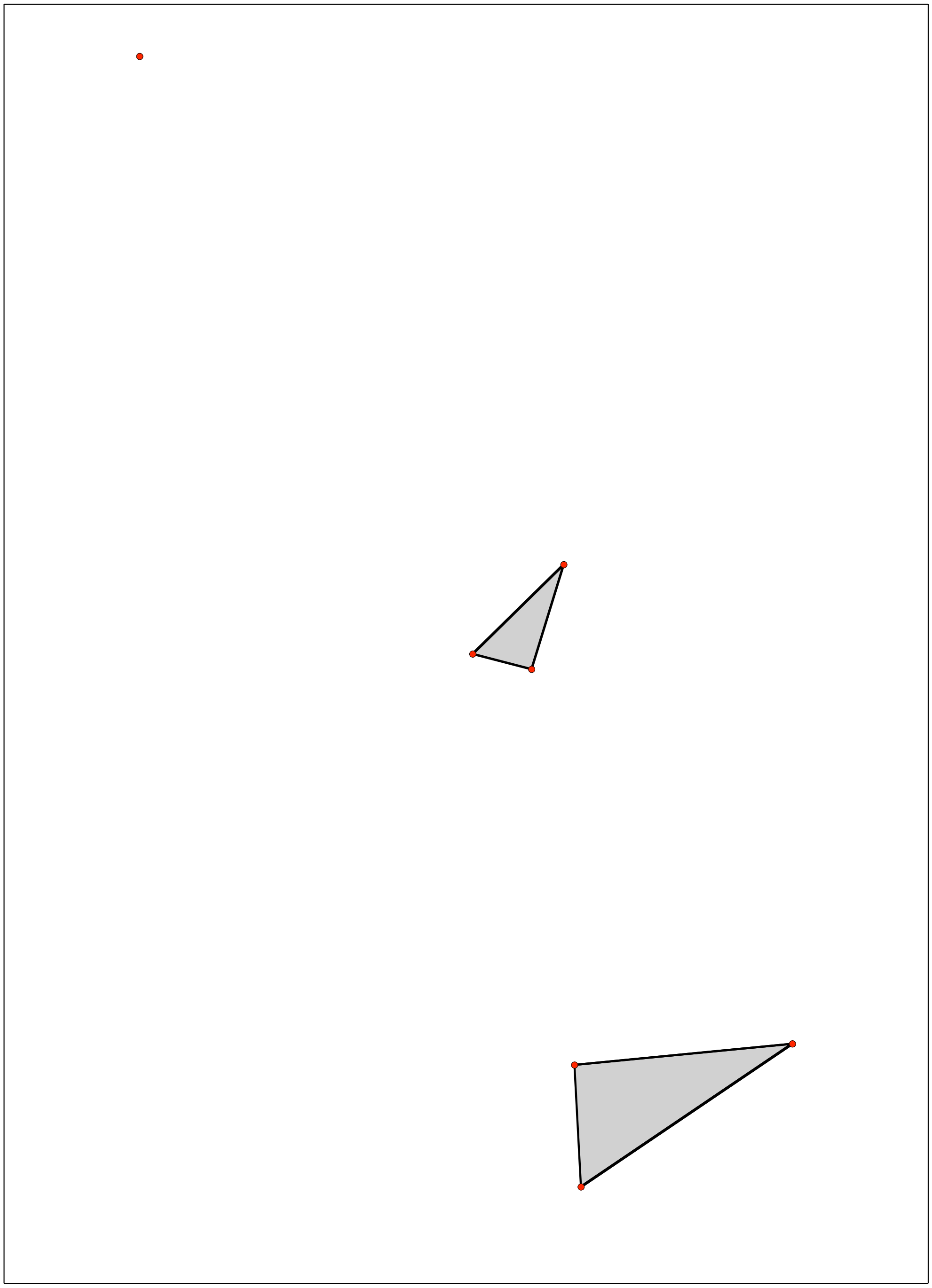


PROOF. A point P is a possible center for the similarity s if and only if $|Ps(X)| = m|PX|$. Let $\mathcal{L}$ denote the locus of points satisfying this equation. If we coordinatize the plane, that equation has the form $\sqrt{(x - x_1)^2 + (y - y_1)^2} = m\sqrt{(x - x_0)^2 + (y - y_0)^2}$, where X has coordinates (x_0, y_0) and $s(X)$ has coordinates (x_1, y_1). Without actually doing the algebra, we see that squaring both sides and collecting terms will result in an equation of a circle. Hence $\mathcal{L}$ is a circle. Two observations: this circle passes through C_1 and C_2 and it is symmetric about the line ℓ. Hence, $\mathcal{L}$ is the circle described above. □

EXERCISE 6.39. *What happens in the above argument if $m = 1$?*

EXERCISE 6.40. *In each of the following two cases identify the similarity that maps the topmost triangle onto the other triangle. Identify the center of the similarity and, if it is direct, compute the angle of rotation, if it is opposite find the axis of reflection.*





CHAPTER 7

Other Matrix linear Function and their Applications

7.1. Systems of Linear Equations

Consider the system of two linear equations in two variables: $ax + by = c$ and $dx + ey = f$. The usual geometric intrepretation of this system is to think of each equation as determining a line and the solution is the intersection of these two lines. We may also think of this system in terms of the the matrix function: $f(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Here, we want to find a point $\begin{bmatrix} x \\ y \end{bmatrix}$ that is mapped onto the point $\begin{bmatrix} c \\ f \end{bmatrix}$. By Theorem **??**, the image of this map is a single point, a line or the entire plane. In the first two cases, it is possible that $\begin{bmatrix} c \\ f \end{bmatrix}$ does not lie in the image of this function and, therefore, there is no solution. In the latter case, the theorem tells us that the function is one-to-one and onto. So in this last case, there is a unique solution. The corollary to the theorem tells us that, if $\Delta\left(\begin{bmatrix} a & b \\ d & e \end{bmatrix}\right) \neq 0$, then the system $ax + by = c$ and $dx + ey = f$ has a unique solution. Actually, we can say more:

Theorem 13. *The system $ax + by = c$ and $dx + ey = f$ has a unique solution if and only if $\Delta\left(\begin{bmatrix} a & b \\ d & e \end{bmatrix}\right) \neq 0$. Furthermore, if $\Delta\left(\begin{bmatrix} a & b \\ d & e \end{bmatrix}\right) = 0$, then there is no solution or a line of solutions or an entire plane of solutions.*

Proof. As we have just pointed out, one direction of the first statement of the theorem follows directly from Theorem **??** and its corollary. The other direction will follow immediately once we prove the second statement. Assume then that $\Delta\left(\begin{bmatrix} a & b \\ d & e \end{bmatrix}\right) = 0$. By Lemma **??**, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} a \\ d \end{bmatrix}$ and $\begin{bmatrix} b \\ e \end{bmatrix}$ are collinear. One possibility is that $\begin{bmatrix} a \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. In this case, $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = y \begin{bmatrix} b \\ e \end{bmatrix}$. Hence, if $\begin{bmatrix} c \\ f \end{bmatrix}$ is in the image, $\begin{bmatrix} c \\ f \end{bmatrix} = y_0 \begin{bmatrix} b \\ e \end{bmatrix}$ and f maps the entire horizontal line $\ell =$

$\left\{\left[\begin{smallmatrix} x \\ y_0 \end{smallmatrix}\right] \mid for\ all\ x\right\}$ onto $\left[\begin{smallmatrix} c \\ f \end{smallmatrix}\right]$. If $\left[\begin{smallmatrix} b \\ e \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, then no points other than those on ℓ are mapped onto $\left[\begin{smallmatrix} c \\ f \end{smallmatrix}\right]$; if $\left[\begin{smallmatrix} b \\ e \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, then $\left[\begin{smallmatrix} c \\ f \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ and the entire plane is mapped onto $\left[\begin{smallmatrix} c \\ f \end{smallmatrix}\right]$.

Finally, assume $\left[\begin{smallmatrix} a \\ d \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. Then $\left[\begin{smallmatrix} b \\ e \end{smallmatrix}\right] = m\left[\begin{smallmatrix} a \\ d \end{smallmatrix}\right]$. So $f\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = (x+ym)\left[\begin{smallmatrix} a \\ d \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} c \\ f \end{smallmatrix}\right]$ is in the image of f if and only if $\left[\begin{smallmatrix} c \\ f \end{smallmatrix}\right] = p\left[\begin{smallmatrix} a \\ d \end{smallmatrix}\right]$, for some p. In this case, f maps the entire horizontal line $\ell = \left\{\left[\begin{smallmatrix} x \\ \frac{p-x}{m} \end{smallmatrix}\right] \mid for\ all\ x\right\}$ onto $\left[\begin{smallmatrix} c \\ f \end{smallmatrix}\right]$. □

EXERCISE 7.1. *Describe the solution set to each of the following systems.*

(i) $3x - 4y = 1$ *and* $-5x + 7y = -1$
(ii) $10x - 4y = 1$ *and* $-5x + 2y = -1$
(iii) $10x - 4y = -6$ *and* $-5x + 2y = 3$

We may solve all all 2 by 2 systems that have a unique solution symbolically.

PROPOSITION 7. *Consider the system* $ax+by = c$ *and* $dx+ey = f$. *If* $\Delta = \Delta\left(\begin{bmatrix} a & b \\ d & e \end{bmatrix}\right) \neq 0$, *the unique solution to this system is given by* $x = \frac{ce-bf}{\Delta}$ *and* $y = \frac{af-cd}{\Delta}$.

PROOF. If $\left[\begin{smallmatrix} a & b \\ d & e \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} c \\ f \end{smallmatrix}\right]$, then $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & b \\ d & e \end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix} c \\ f \end{smallmatrix}\right]$ By Lemma **??** (iii), $\left[\begin{smallmatrix} a & b \\ d & e \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} \frac{e}{\Delta} & -\frac{b}{\Delta} \\ -\frac{d}{\Delta} & \frac{a}{\Delta} \end{smallmatrix}\right]$. Then by multiplying out $\left[\begin{smallmatrix} \frac{e}{\Delta} & -\frac{b}{\Delta} \\ -\frac{d}{\Delta} & \frac{a}{\Delta} \end{smallmatrix}\right]\left[\begin{smallmatrix} c \\ f \end{smallmatrix}\right]$ gives the solution recorded above. □

EXERCISE 7.2. *Verify by direct computation that, when* $\Delta = \Delta\left(\left[\begin{smallmatrix} a & b \\ d & e \end{smallmatrix}\right]\right) \neq 0$, *then* $x = \frac{ce-bf}{\Delta}$ *and* $y = \frac{af-cd}{\Delta}$. *is a solution to the system* $ax + by = c$ *and* $dx + ey = f$.

Looking carefully at this formula for the solution, we see that the numerators are also determinants: $ce - bf = \Delta\left(\left[\begin{smallmatrix} c & b \\ f & e \end{smallmatrix}\right]\right)$ and $af - cd = \Delta\left(\left[\begin{smallmatrix} a & c \\ d & f \end{smallmatrix}\right]\right)$. This formulation is called *Cramer's Rule* and works for a system of n equations in n unknowns.

THEOREM 14. *Consider the system:*

$$\begin{array}{ccccc} m_{11}x_1 + m_{12}x_2 & \dots & m_{1n}x_n = b_1 \\ \vdots \quad \vdots & \vdots & \vdots \quad \vdots \\ m_{n1}x_1 + m_{n2}x_2 & \dots & m_{nn}x_n = b_n \end{array}.$$

If $\Delta = \Delta\left(\begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix}\right) \neq 0$*, then* $x_i = \frac{\Delta_i}{\Delta}$*, for all* i*, is the unique solution to this system, where* Δ_i *is the determinant of the matrix obtained from* $\begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix}$ *by replacing the ith column with* $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

EXERCISE 7.3. *(3D) Use the formula for the determinant of a 3 by 3 matrix to verify Cramer's formula for systems of 3 equations in 3 unknowns.*

7.2. Input-Output Models

This section is based on the work of Wassily Leontief for which he won the 1973 Nobel Prize in Economics. Suppose that we have a small country with a simple economy having just two sectors, agriculture and manufacturing. Each sector produces commodities that are essential to their own sector and to the other sector and each produces commodities that are sold outside of these two sectors. For example, the agriculture produces seed and natural fertilizer to be used by the agriculture sector, cotton and peanuts to be processed into clothing and peanut butter by the manufacturing sector and fruits and vegetables that are sold directly to the consumer. Each sector of this simple economy produces at some level that can be reported at a single dollar amount. For example, agriculture could be producing at the level of \$75,000,000 per year and manufacturing at the level of \$62,000,000 per year. Next suppose that, for each dollar of agriculture output, we use .32 dollars worth of input from the agriculture sector (seed, natural fertilizer, etc.) and .31 dollars worth of input from the manufacturing sector (tractors, pesticides, etc.) and that, for each dollar of manufacturing output we use .57 dollars worth of input from the agriculture sector (cotton, peanuts, etc.) and .45 dollars worth of input from the manufacturing sector (steel, tools used in manufacturing, etc.). We can now do some accounting: of the \$75,000,000 of agriculture production \$24,000,000 is used by agriculture (.32×$75,000,000) and $35,340,000 is used by the manufacturing sector (.57×$62,000,000) this leaves $15,660,000 of the agriculture output to be sold to the general public:

$$75,000,000 - .32 \times 75,000,000 - .57 \times 62,000,000 = 15,660,000.$$

The accounting for the manufacturing sector is similar:

$$62,000,000 - .31 \times 75,000,000 - .45 \times 62,000,000 = 10,850,000.$$

We may represent these equations in matrix form:

$$\begin{bmatrix} 75 \\ 62 \end{bmatrix} - \begin{bmatrix} .32 & .57 \\ .31 & .45 \end{bmatrix} \begin{bmatrix} 75 \\ 62 \end{bmatrix} = \begin{bmatrix} 15.66 \\ 10.85 \end{bmatrix},$$

where production levels are reported in millions of dollars.

The advantage of this matrix formulation is that it separates the production levels and final output (the column vectors) from dependency structure of these two segments of the economy (the 2 by 2 matrix). In this form, the economist can easily use this model to study the effects of changes in the demand for agricultural and manufactured products. Suppose that due to crop failures elsewhere, the demand for agricultural products increased from 15.66 million dollars to 20 million. How would the country's economy adjust to meet this demand? Just increasing agriculture production by 4.34 million won't work since it does not take into account the additional input from agriculture and manufacturing needed to increase agriculture output. What we need to do is to solve the following matrix equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} .32 & .57 \\ .31 & .45 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 20 \\ 10.85 \end{bmatrix},$$

or simply $X - MX = D$, where $M = \begin{bmatrix} .32 & .57 \\ .31 & .45 \end{bmatrix}$, $D = \begin{bmatrix} 20 \\ 10.85 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$ denoting the yet to be determined production levels x and y for agriculture and manufacturing, respectively. M is called the —- matrix and D the *demand vector.* This matrix equation is easily solved, $X = (I - M)^{-1}D$, and we note that the solution is the fixed point of the matrix linear function $f(X) = MX + D$. The matrix $(I - M)^{-1}$ is called the *Leontief inverse* in economics. Computing X in this case, we see that to meet the increased demand for agricultural products, the agriculture segment must increase it output to 87.1 million per year and the manufacturing segment must increase it output to 68.82 million per year.

EXERCISE 7.4. *Compute the production levels for the above simple economy needed to meet a demand of 20 million from each sector. What* *if there was a demand of 20 million for agricultural products but no demand for manufactured products?*

Of course economies are much more complicated. The initial model of the US economy constructed by Leontief used 500 sectors. Hence, he was faced with computing the Leontief inverse of a 500 by 500 matrix. Given the state of computation technology at the time this was a major task.

7.3. Affine Geometry

We may think of "affine geometry" as the study of the geometric properties preserved by affinities. Length or distance is not an affine property: an affinity may stretch some lengths and shrink others. Nevertheless, all segments are stretch or shrink linearly. As we have noted, area is an affine property in that all areas are multiplied by the same factor under an affinity. Angle measure is not an affine property: an affinity that is not a similarity will stretch some some angle measures and shrinking others. Parallelism is an affine property: parallel lines will be mapped onto parallel lines.

EXERCISE 7.5. *Consider the affinity* $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. *Let* $O = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $P = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and* $Q = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$. *Compute the lengths of the segments* OP, OQ *and* PQ *as well as the length of their images under the given affinity. Also verify that the midpoint of each segment is mapped onto the midpoint of its image segment.*

To expand on the opening paragraph of this section, reconsider Theorem **??**, Lemma **??** and Theorem **??**. Theorem **??** tells us that, under an affinity, all areas are magnified by the same positive real number and all orientations preserved or all orientations are reversed. The lemma tells us that an affinity f maps a line ℓ onto a line $f(\ell)$, magnifying all lengths by the same factor $\frac{|f(P_0)f(P_1)|}{|P_0P_1|}$, where P_0 and P_1 are any two points on ℓ. It follows at once that f maps the midpoint of the segment P_0P_1 onto the midpoint of the segment $f(P_0)f(P_1)$. In fact if $X_0 = P_0, X_1, \ldots, X_n = P_1$ divide P_0P_1 into n segments of equal length, then $f(X_0), f(X_1), \ldots, f(X_n)$ divide $f(P_0)f(P_1)$ into n segments of equal length. Theorem **??** (iv) tells us that parallel lines are mapped onto parallel lines. Hence parallelograms are mapped onto parallelograms. Since opposite sides of a parallelogram are equal, it follows that an affinity magnifies parallel lines by the same factor.

EXERCISE 7.6. *Write out the proof that*

(i) *parallelograms are mapped onto parallelograms and*
(ii) *an affinity magnifies parallel lines by the same factor.*

For euclidean geometry, we defined two figures to be congruent if there was a congruence that mapped one onto the other and two figures to be similar if there was a similarity that mapped one onto the other. We now define two figures to be *affinely equivalent* if one is the image of the other under an affinity.

PROPOSITION 8.

(i) *All (non-degenerate) triangles are affinely equivalent.*
(ii) *All (non-degenerate) parallelograms are affinely equivalent.*

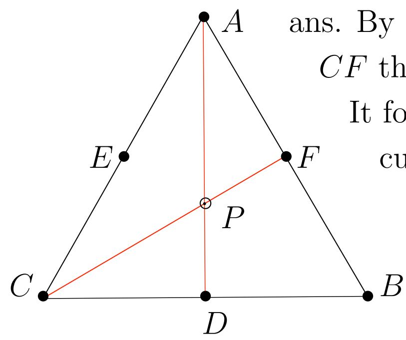
PROOF. Let P_0, P_1 and P_2 and Q_0, Q_1 and Q_2 be the vertices of non-degenerate (positive area) triangles $\triangle P_0P_1P_2$ and $\triangle Q_0Q_1Q_2$, respectively. It follows that P_0, P_1 and P_2 and Q_0, Q_1 and Q_2 are both non-collinear triples. Let f denote the matrix linear function, guaranteed by Lemma **??**, that maps $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ onto P_0, P_1 and P_2, respectively, and let g denote the matrix linear function that maps $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ onto Q_0, Q_1 and Q_2, respectively. Since these sets are non-collinear the images of these matrix linear functions are not lines or points. Hence f and g are affinities. By the corollary to Theorem **??**, f^{-1} is a similarity. It follows from Lemma **??** that the composition gf^{-1} is an affinity.

Now consider the parallelograms with vertices (labeled consecutively around each figure) P_0, P_1, P_2 and P_3 onto Q_0, Q_1, Q_2 and Q_3, respectively. Let f be the affinity that maps $\triangle P_0P_1P_2$ onto $\triangle Q_0Q_1Q_2$. Note that P_3 is the intersection of the line through P_0, parallel to P_1P_2 and the line through P_2, parallel to P_0P_1. Similarly, Q_3 is the intersection of the line through Q_0, parallel to Q_1Q_2 and the line through Q_2, parallel to Q_0Q_1. Since f is an affinity, it maps the line through P_0, parallel to P_1P_2, onto the line through Q_0, parallel to Q_1Q_2 and the line through P_2, parallel to P_0P_1 onto the line through Q_2, parallel to Q_0Q_1. Therefore, f must map P_3 onto Q_3. □

Some theorems in euclidian geometry are really "affine" theorems. For example:

PROPOSITION 9. *The medians of a triangle are concurrent and the common point divides each median in the ratio 1 to 2.*

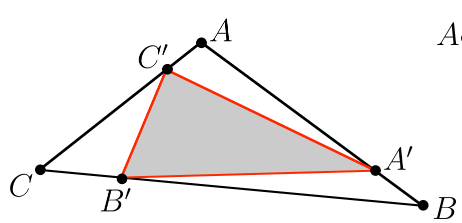
PROOF. Consider an equilateral triangle and construct two medi-

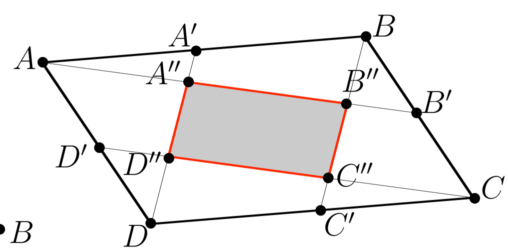


ans. By symmetry, the reflection of the median CF through the median AD is the median BE. It follows at once that the medians are concurrent. By symmetry the area of $\triangle CPB$ is one third of the area of $\triangle CAB$. Since these triangles have the same base, PD, the hight of $\triangle CPB$, is one third of AD, the hight of $\triangle CAB$. Now consider and arbitrary triangle $\triangle A'B'C'$. The affinity that maps $\triangle ABC$ onto $\triangle A'B'C'$ maps the medians of $\triangle ABC$ onto medians of $\triangle A'B'C'$, the point of concurrence of the medians of $\triangle ABC$ onto a point of concurrence of the medians of $\triangle A'B'C'$ and preserves the $\frac{1}{3}$-$\frac{2}{3}$ partition of the medians. $\square$

In general any theorem about triangles or parallelograms that involve only affine properties need only be proved in the special case of the equilateral triangle or the square. An affinity then maps this special case onto an arbitrary triangle or parallelogram, preserving the relationships between affine properties. We state several of these affine results, leaving the proofs as exercises for the reader.

Proposition 10. *Let r be a real number between 0 and 1, let $\triangle ABC$ be any triangle. Let A' be the point on AB so that $|AA'| = r|AB|$, let B' be the point on BC so that $|BB'| = r|BC|$ and C' be the point on CA so that $|CC'| = r|CA|$. Then the area of $\triangle A'B'C'$ is $(1 - 3r + 3r^2)$ times the area of $\triangle ABC$.*





Proposition 11. *Let r be a real number between 0 and 1, let $\triangle ABCD$ be any parollelogram. Let A' be the point on AB so that $|AA'| = r|AB|$, let B' be the point on BC so that $|BB'| = r|BC|$, let C' be the point on CD so that $|CC'| = r|Cd|$ and D' be the point on DA so that $|DD'| = r|dA|$. Now let A'' be the intersection of the segments DA' and AB', let B'' be the intersection of the segments AB' and BC', let C'' be the intersection of the segments BC' and CD' and let D''*

be the intersection of the segments CD' and DA'. Then $A''B''C''D''$ is a parallelogram with area $\frac{(1-r)^2}{1+r^2}$ times the area of the parallelogram $ABCD$.

Exercise 7.7. *Prove Proposition* **??**.

Exercise 7.8. *Prove Proposition* **??**.

Exercise 7.9. *State and prove an analogue to Proposition* **??** *for parallelograms.*

Exercise 7.10. *Prove that "affinely equivalent" is an equivalence relation.*

Exercise 7.11. *A quadrilateral with two opposite sides parallel is called a trapezoid. Describe the affine equivalence classes of trapezoids.*

7.4. Affinities

The "affinities" of the line were all similarities and therefore it was easy to describe just how they mapped the line onto itself. Similarly, it is easy to describe the action of the similarities of the plane. But, for the application we will discuss later, we will need to know just how the remaining affinities map the plane onto itself. Affinities naturally fall into two groupings: those that have a fixed point and those that do not. We start our investigation by considering affinities that have a fixed point.

Let $f(X) = MX + B$ and recall that f has a fixed point C if and only if $I - M$ has an inverse; then $C = (I - M)^{-1}B$. In the case that f has a fixed point, we may rewrite f in the form:

$$f(X) = MX + B = MX + (I - M)C = M(X - C) + C.$$

Skipping the intermediate steps, we have $f(X) = M(X-C)+C$. Thus, we may interpret the action of f as follows: translate the plane by $-C$, that is translate C to the origin, then multiply by the matrix M and finally translate back by the vector C. We can state this conclusion as follows:

If $f(X) = MX + B$ has a fixed point C, the action of f on the plane about its center C is the same as the action of $g(X) = MX$ about the origin.

Hence to understand the action of affinities with fixed points we need only understand the action of affinities of the form $f(X) = MX$.

One of the most useful tools in investigating the general matrix function $f(X) = MX$ is the concept of eigenvector: a non-zero vector

X is an *eigenvector* for M if $MX = \lambda X$, for some real number λ, called an *eigenvalue* for M. Suppose that X is an eigenvector for M with eigenvalue λ. Then $(M - \lambda I)X = Z$, where Z is the zero vector or origin. Hence, if λ is an eigenvalue, the matrix function $(M - \lambda I)X$ is not one to one and hence not an affinity. In more practical terms, λ is an eigenvalue for M if and only if the determinant $\Delta(M - \lambda I)$ is zero. Therefore, the eigenvalues of $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} X$ are the roots of the quadratic equation $\Delta(M - \lambda I) = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + \Delta M = 0$. The sum of the entries on the main diagonal of M ($a + d$ in this case) is called the *trace* of M and the polynomial $\lambda^2 - T\lambda + \Delta$, where T and Δ are, respectively, the trace and determinant of M, is called the *characteristic polynomial* of M. We have proved the first part of:

LEMMA 36. *Let M be a 2 by 2 matrix. Then*

(i) *The eigenvalues of M are the roots of its characteristic polynomial.*

(ii) *If M admits eigenvalues λ_1 and λ_2, perhaps the same, then the trace of M is $\lambda_1 + \lambda_2$ and the determinant of M is $\lambda_1\lambda_2$.*

PROOF. Consider the quadratic polynomial $p(\lambda) = \lambda^2 + T\lambda + \Delta$. We recall first that, if $p(\lambda)$ does not factor, then $p(\lambda)$ has no real roots and second that, if it does factor $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$, then $p(\lambda)$ has exactly two roots, λ_1 and λ_2 (which may be equal). Expanding $(\lambda - \lambda_1)(\lambda - \lambda_2)$ gives $p(\lambda) = \lambda^2 + (\lambda_1 + \lambda_2)\lambda + (\lambda_1\lambda_2)$. □

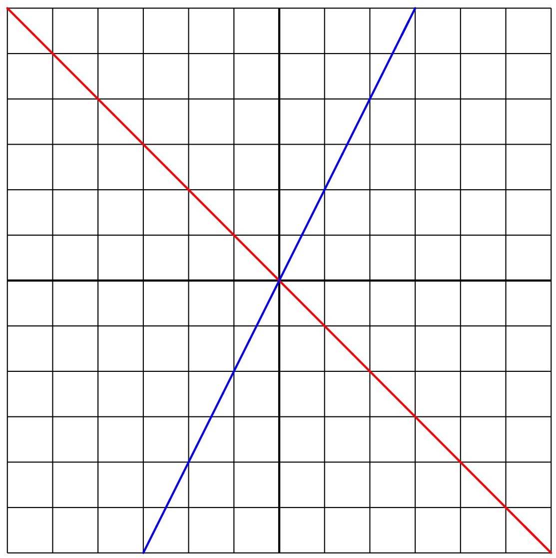
Let's work through a simple example, $f(X) = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} X$. Computing the characteristic polynomial $\Delta\left(\begin{bmatrix} 0 - \lambda & 1 \\ 2 & 1 - \lambda \end{bmatrix}\right)$ and setting it equal to zero gives the quadratic equation $\lambda^2 - \lambda - 2 = 0$. Of course, we could have concluded that $T = 1$ and $\Delta = 2$ by inspection and concluded directly that $\lambda^2 - \lambda - 2$ was the characteristic polynomial. Factoring, we have $(\lambda + 1)(\lambda - 2) = 0$, giving the eigenvalues -1 and 2. Next we have to find the eigenvectors that go with each of these eigenvalues. We do this by finding non-zero solutions to the following two matrix equations:

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix} \textit{ and } \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

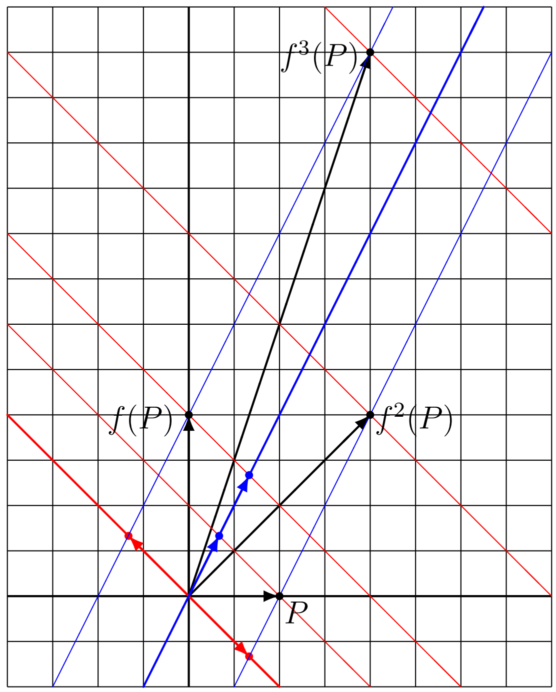
These gives rise to a redundant system of equations for each eigenvalue:

$$\left\{ \begin{array}{ccc} y & = & -x \\ 2x + y & = & -y \end{array} \right\} \textit{ and } \left\{ \begin{array}{ccc} y & = & 2x \\ 2x + y & = & 2y \end{array} \right\}.$$

Solving, we see that the vectors of the form $\begin{bmatrix} a \\ -a \end{bmatrix}$ are all eigenvectors for -1 and the vectors of the form $\begin{bmatrix} a \\ 2a \end{bmatrix}$ are all eigenvectors for 2. So the eigenvectors of -1 form the line $y = -x$ and the eigenvectors of 2 form the line $y = 2x$. To interpret this geometrically, consider the coordinate plane pictured below. The eigenvectors of -1 are shown in red and the eigenvectors of 2 are shown in blue.



Each vector on the blue line is doubled in length by f: $f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $f\left(\begin{bmatrix} 1.5 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, $f\left(\begin{bmatrix} -.5 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ and so on. In other words, the restriction of f to the blue line is the dilation by a factor of 2 with the origin as center. If we restrict f to the red line, we have the reflection about the origin: $f\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $f\left(\begin{bmatrix} -1.5 \\ 1.5 \end{bmatrix}\right) = \begin{bmatrix} 1.5 \\ -1.5 \end{bmatrix}$ and so on. To see how this transformation maps a point not on one of these lines, we must resolve it, that is write it as the sum of a vector on each line. To do this for a vector X, draw the lines through X parallel to the red and blue lines; X will then be the sum of the vectors corresponding to the intersections. We illustrate this with the point $P = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ on the x-axis.



To understand just how f maps $P = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ we write P as a sum of eigenvectors: $\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ -\frac{4}{3} \end{bmatrix} + \begin{bmatrix} \frac{2}{3} \\ \frac{4}{3} \end{bmatrix}$. Then

$$f\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = f\left(\begin{bmatrix} \frac{4}{3} \\ -\frac{4}{3} \end{bmatrix}\right) + f\left(\begin{bmatrix} \frac{2}{3} \\ \frac{4}{3} \end{bmatrix}\right) = \begin{bmatrix} -\frac{4}{3} \\ \frac{4}{3} \end{bmatrix} + \begin{bmatrix} \frac{4}{3} \\ \frac{8}{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

Using the decomposition already computed for $\begin{bmatrix} 0 \\ 4 \end{bmatrix}$, we see that $f^2\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = f\left(\begin{bmatrix} 0 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ and then $f^3\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = f\left(\begin{bmatrix} 4 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$. We can describe the action of f as follows $f(X)$ is constructed by "flipping" X over the line of eigenvectors $\begin{bmatrix} b \\ 2b \end{bmatrix}$ (the thick blue line) and doubling its distance from the line of eigenvectors $\begin{bmatrix} a \\ -a \end{bmatrix}$ (the thick red line). Note that since the two lines of eigenvectors are not perpendicular, this "flipping" is not a reflection.

Exercise 7.12. *Compute the eigenvectors for each of the following matrices, M, and describe the action of the associated matrix equation $f(X) = MX$.*

(i) $M = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$

(ii) $M = \begin{bmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{bmatrix}$

(iii) $M = \begin{bmatrix} \frac{5}{2} & 2 \\ -1 & -\frac{1}{2} \end{bmatrix}$

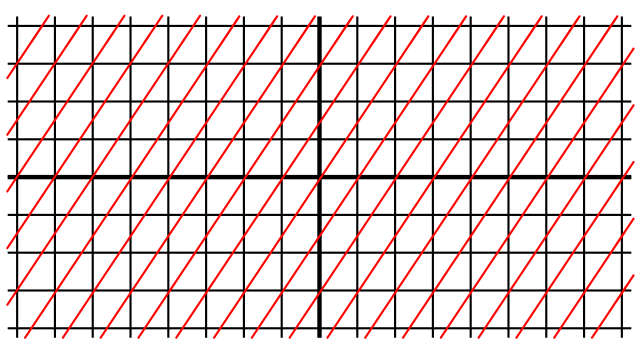
Next suppose that the characteristic polynomial of M has a double root then there are two possibilities: there are two "independent" eigenvectors or just one line of eigenvectors. Let X_1 and X_2 be two eigenvectors of M with the same eigenvalue λ and assume that X_2 is not a scaler multiple of X_1. Then any vector X can be written in the form $a_1X_1 + a_2X_2$ for some real numbers a_1 and a_2. It follows that X is also an eigenvector for M with eigenvalue λ:

$$\begin{aligned} MX &= M(a_1X_1 + a_2X_2) \\ &= a_1MX_1 + a_2MX_2 \\ &= a_1\lambda X_1 + a_2\lambda X_2 \\ &= \lambda(a_1X_1 + a_2X_2) = \lambda X. \end{aligned}$$

If $\lambda > 0$, then MX is simply a dilation with magnification λ and, when $\lambda < 0$, MX is a rotor-dilation with magnification $|\lambda|$ and a half turn.

In both cases, since $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are both eigenvectors, $M = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$.

In the case that the characteristic polynomial of M has a double root but there is only one line of eigenvectors can best be illustrated by assuming that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector. Then $M = \begin{bmatrix} \lambda & m \\ 0 & \lambda \end{bmatrix}$. In the simplest case $\lambda = 1$ we have a *shear* : the x-axis is fixed points a distance d above the x-axis are shifted $|m|d$ units to the right (to the left if $m < 0$) while points a distance d below the x-axis are shifted $|m|d$ units to the left (to the right if $m < 0$). The distortion of the unit grid is shown in the following picture: the x-axis is point wise fixes all other horizontal lines are mapped onto themselves; the images of the vertical lines are shown in red.



In general, we can rewrite M as a shear followed by a dilation: $M = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} 1 & \frac{m}{\lambda} \\ 0 & 1 \end{bmatrix}$.

If the characteristic polynomial of M has a double root but there is only one line of eigenvectors and that line is not the x-axis, we simply rotate the above picture.

EXERCISE 7.13. *Compute the eigenvectors for each of the following matrices, M, and describe the action of the associated matrix equation $f(X) = MX$.*

(i) $M = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$
(ii) $M = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$
(iii) $M = \begin{bmatrix} 1 & -\frac{3}{2} \\ \frac{3}{2} & -2 \end{bmatrix}$

It would be nice to be able simply look at a matrix and decide if it has two one or no eigenvalues. To this end we prove the following useful lemma.

LEMMA 37. *Consider the matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (with trace $T = a+d$ and determinant $\Delta = ad - bc$). Then:*

(i) *M has two distinct eigenvalues when $T^2 > 4\Delta$ or equivalently, $bc > -(\frac{a-d}{2})^2$; in this case the eigenvalues are $\frac{1}{2}(a+d) \pm \frac{1}{2}\sqrt{(a-d)^2 + 4bc}$.*
(ii) *M has exactly one eigenvalue when $T^2 = 4\Delta$ or equivalently, $bc = -(\frac{a-d}{2})^2$; in this case that eigenvalue is $\frac{1}{2}(a+d)$;*
(iii) *M has no eigenvalues when $T^2 < 4\Delta$ or equivalently, $bc < -(\frac{a-d}{2})^2$.*

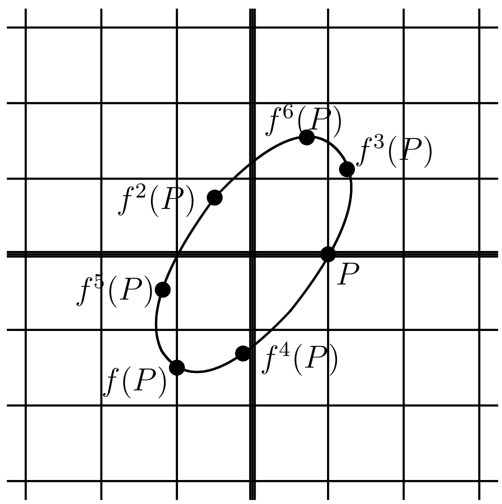
PROOF. Recall that the characteristic equation for M is $\lambda^2 - T\lambda + \Delta = 0$, where $T = a + d$ and $\Delta = ad - bc$. From the quadratic formula we have that the eigenvalues of M are given by $\lambda = \frac{1}{2}T \pm \frac{1}{2}\sqrt{T^2 - 4\Delta}$. Expanding the decrement: $T^2 - 4\Delta = (a+d)^2 - 4ad + 4bc = (a-d)^2 + 4bc$. So the eigenvalues are given by$(a+d)^2 \pm \frac{1}{2}\sqrt{(a-d)^2 + 4bc}$ and the result follows. □

EXERCISE 7.14. *In each case construct a 2 by 2 matrix satisfying the given set of conditions.*

(i) *M has top row (1,1) and eigenvalues 2 and -2*
(ii) *M has top row (1,1) and the single eigenvalue -2*
(iii) *M has top row (0,1) and no eigenvalues*

Finally, we consider the case that M has no eigenvalues. We start by considering a example: $M = \begin{bmatrix} -1 & 1 \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$. One easily checks that the determinant of this matrix is 1 and the trace is $-\frac{1}{2}$ and that condition (iii) of the lemma holds. So there are no eigenvalues nor eigenvectors for this matrix. How then can we picture the action of this matrix? Let's follow the trajectory of the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ under repeated iterations of this transformation: $\begin{bmatrix} -1 & 1 \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{3}{2} \end{bmatrix}$, $\begin{bmatrix} -1 & 1 \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 \\ -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{4} \end{bmatrix}$, and so on. The result of the first 6 iterations is listed below and these point are then plotted.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -\frac{3}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{4} \end{bmatrix}, \begin{bmatrix} \frac{5}{4} \\ \frac{9}{8} \end{bmatrix}, \begin{bmatrix} -\frac{1}{8} \\ -\frac{21}{16} \end{bmatrix}, \begin{bmatrix} -\frac{19}{16} \\ -\frac{15}{32} \end{bmatrix}, \begin{bmatrix} \frac{23}{32} \\ \frac{99}{64} \end{bmatrix} \cdots$$



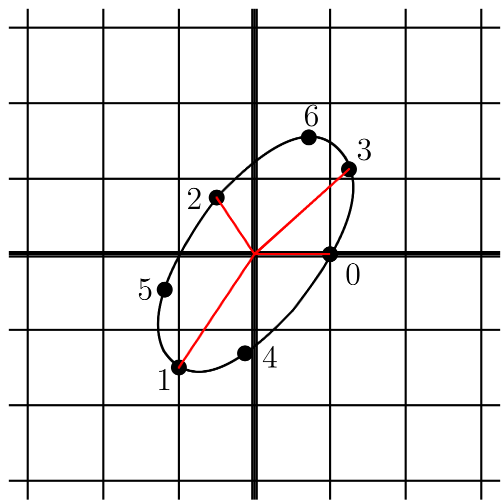
These point appear to lie on a ellipse and in fact they do. Note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ lies on the ellipse $3x^2 - 3xy + 2y^2 = 3$ and one can easily check that each of the points in the iteration of the transformation starting with $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ also lies on this ellipse. In fact we can prove that, if $\begin{bmatrix} x \\ y \end{bmatrix}$ lies on the ellipse $3x^2 - 3xy + 2y^2 = c$, then its image $\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}$ under multiplication by M is also on this ellipse. Computing the image, we have $\overline{x} = (-x + y)$ and $\overline{y} = (-\frac{3}{2}x + \frac{1}{2}y)$. Expanding $3\overline{x}^2 - 3\overline{xy} + 2\overline{y}^2$, we get:

$$3(-x+y)^2 - 3(-x+y)(-\frac{3}{2}x+\frac{1}{2}y) + 2(-\frac{3}{2}x+\frac{1}{2}y)^2 = 3x^2 - 3xy + 2y^2 = c.$$

So, each ellipse in the family of ellipses $\{3x^2 - 3xy + 2y^2 = c\}$ is mapped onto itself by the transformation $f(X) = MX$ and the trajectory of any point X under repeated application of f is a sequence of points on the ellipse in the family that contains X.

Now consider the ellipse containing $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the sector of that ellipse bounded by the segment from the origin to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the segment from the origin to $\begin{bmatrix} -1 \\ -\frac{3}{2} \end{bmatrix}$ and the arc of the ellipse joining $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and

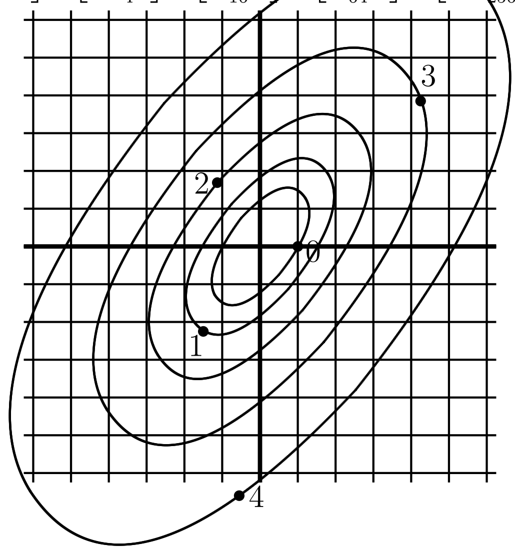
$\begin{bmatrix} -1 \\ -\frac{3}{2} \end{bmatrix}$. Under one application of f, this sector is mapped onto the sector bounded by the segment from the origin to $\begin{bmatrix} -1 \\ -\frac{3}{2} \end{bmatrix}$, the segment from the origin to $\begin{bmatrix} -\frac{1}{2} \\ \frac{3}{4} \end{bmatrix}$ and the arc of the ellipse joining $\begin{bmatrix} -1 \\ -\frac{3}{2} \end{bmatrix}$ and $\begin{bmatrix} -\frac{1}{2} \\ \frac{3}{4} \end{bmatrix}$. Since the determinant of M is 1, these two sectors must have the same area! Indeed all of the sectors defined by the images of two successive iterations have the same area.



Exercise 7.15. *Look up "Kepler's Second Law of planetary motion" and relate it to this last observation.*

Suppose we multiply M by the dilation $\begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}$; the resulting matrix is $M^* = \begin{bmatrix} -d & d \\ -\frac{3}{2}d & \frac{1}{2}d \end{bmatrix}$. The trace of this matrix is d times the trace of M, the determinant of the is matrix is d^2 and hence the characteristic polynomial of this matrix is $\lambda^2 - \frac{d}{2}\lambda + d^2$. One easily checks that this matrix, like M, has no eigenvalues. It does have the same family of ellipses attached to it. However, if X lies on the ellipse $3x^2 - 3xy + 2y^2 = c$, then M^*X lies on the ellipse $3x^2 - 3xy + 2y^2 = d^2c$. We illustrate this with the case $d = \frac{3}{2}$ and $M^* = \begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ -\frac{9}{4} & \frac{3}{4} \end{bmatrix}$. Multiplying $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ on $3x^2 - 3xy + 2y^2 = 3$ by M^* yields $\begin{bmatrix} -\frac{3}{2} \\ -\frac{9}{4} \end{bmatrix}$ on $3x^2 - 3xy + 2y^2 = \frac{27}{4}$; multiplying by M^* again results in $\begin{bmatrix} -\frac{9}{8} \\ \frac{27}{16} \end{bmatrix}$ on $3x^2 - 3xy + 2y^2 = \frac{234}{16}$ and so on as pictured below. So we may think of the iterates of X as moving out on an elliptical spiral. If $\Delta = d^2$ were less than one they would spiral in.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{2} \\ -\frac{9}{4} \end{bmatrix}, \begin{bmatrix} -\frac{9}{8} \\ \frac{27}{16} \end{bmatrix}, \begin{bmatrix} \frac{135}{32} \\ \frac{243}{64} \end{bmatrix}, \begin{bmatrix} -\frac{81}{128} \\ -\frac{1701}{256} \end{bmatrix} \cdots$$



There is a family of conic sections associated with every matrix M. We start by looking at those matrices that have trace T, determinant $\Delta = \pm 1$ and are oriented so that they have the form $M = \begin{bmatrix} \frac{T}{2} & a \\ \frac{T^2 \mp 4}{4a} & \frac{T}{2} \end{bmatrix}$, where $a \neq 0$. In this orientation the associated conics are symmetric about the coordinate axes: by direct computation, we see that if $\begin{bmatrix} x \\ y \end{bmatrix}$ lies on the conic $(T^2 \pm 4)x^2 + 4a^2y^2 = c$ then so does $M \begin{bmatrix} x \\ y \end{bmatrix}$. The characteristic equation for this matrix is $\lambda = \frac{T}{2} \pm \sqrt{\frac{T^2}{4} - 1}$. There are four cases to consider.

In the case $\Delta = 1$ and $|T| < 2$ there are no real eigenvalues and the conic is an ellipse. This is just like the above example.

In the case $\Delta = 1$ and $|T| = 2$, there is just one eigenvalue $\frac{T}{2} = 1$ or $\frac{T}{2} = -1$ with the x axis as the line of eigenvectors. The associated conics are the degenerate conics consisting of two lines $y = c$ and $y = -c$, for some constant c.

In the case $\Delta = 1$ and $|T| > 2$, there are two distinct eigenvalue $\frac{T}{2} \pm \frac{1}{2}\sqrt{T^2 - 4}$ and one easily checks that the lines of eigenvectors are $y = \pm \frac{T^2-4}{2a}x$. The associated conics are are the hyperbolas $(4 - T^2)x^2 + 4a^2y^2 = c$ with the lines of eigenvectors as asymptotes.

Finally, in the case $\Delta = -1$, there are two distinct eigenvalue $\frac{T}{2} \pm \frac{1}{2}\sqrt{T^2 + 4}$ and one easily checks that the lines of eigenvectors are $y = \pm \frac{T^2+4}{2a}x$. The associated conics are are the hyperbolas $-(T^2 + 4)x^2 +$

$4a^2y^2 = c$ with the lines of eigenvectors as asymptotes. We restate these results as a lemma for later reference:

LEMMA 38. *Let* $M = \begin{bmatrix} \frac{T}{2} & a \\ \frac{T^2-\Delta 4}{4a} & \frac{T}{2} \end{bmatrix}$, *where* $a \neq 0$. *Then the characteristic polynomial of* M *is* $\lambda^2 - T\lambda + \Delta$, *giving* $\lambda = \frac{T}{2} \pm \frac{1}{2}\sqrt{T^2 - 4\Delta}$. *There are four cases:*

(i) $\Delta = 1$ *and* $|T| < 2$. *In this case,* M *has no eigenvalues and the associated conics are ellipses of the form* $(4 - T^2)x^2 + 4a^2y^2 = c$.
(ii) $\Delta = 1$ *and* $|T| = 2$. *In this case,* M *has exactly one eigenvalue, 1 or -1, and one line of eigenvectors, the* x*-axis; the associated conics are pairs of straight horizontal lines,* $y = \pm c$.
(iii) $\Delta = 1$ *and* $|T| > 2$. *In this case,* M *has two distinct eigenvalues* $\frac{T}{2} \pm \frac{\sqrt{T^2-4}}{2}$ *with two lines of eigenvectors* $y = \pm\frac{\sqrt{T^2-4}}{2a}x$; *the associated conics have equations of the form* $(4-T^2)x^2 + 4a^2y^2 = c$ *and are hyperbolas with the lines of eigenvectors as asymptotes.*
(iv) $\Delta = -1$. *In this case,* M *has two distinct eigenvalues* $\frac{T}{2} \pm \frac{\sqrt{T^2+4}}{2}$ *with two lines of eigenvectors* $y = \pm\frac{\sqrt{T^2+4}}{2a}x$; *the associated conics have equations of the form* $(T^2+4)x^2 - 4a^2y^2 = c$ *and are hyperbolas with the lines of eigenvectors as asymptotes.*

In each first three cases, the iterates of X_0, $\{X_i = MX_{i-1}\}$ *move along the conic containing* X_0 *so that the areas of successive sectors (bounded by the segments* OX_i *and* OX_{i+1} *and the corresponding arc of the conic) all have the same area. In the fourth case, the even iterates of* X_0 *(*$X_0, X_2, X_4, \ldots$*) move along the conic containing* X_0 *so that the areas of successive sectors (bounded by the segments* OX_i *and* OX_{i+1} *and the corresponding arc of the conic) all have the same area. Similarly, the odd iterates of* X_0 *(*$X_1, X_3, X_5, \ldots$*) move along the conic containing* X_0 *so that the areas of successive sectors (bounded by the segments* OX_i *and* OX_{i+1} *and the corresponding arc of the conic) all have the same area.*

PROOF. Only the last conclusion remains to be proved. The key here is to observe that M^2 has determinant 1. By direct computation $M^2 = \begin{bmatrix} \frac{T^2+2}{2} & aT \\ \frac{(T^2+4)T}{4a} & \frac{T^2+2}{2} \end{bmatrix} = \begin{bmatrix} \frac{\overline{T}}{2} & \overline{a} \\ \frac{\overline{T}^2-4}{4\overline{a}} & \frac{\overline{T}}{2} \end{bmatrix}$, where $\overline{a} = aT$ and $\overline{T} = T^2 + 2$. Hence, M^2 has the standard form. Furthermore it is easy to see that M^2 has the squares of the eigenvalues of M as its eigenvalues and that

M^2 has the same lines of eigenvectors as M. Therefore, M^2 has the same associated family of conics. □

We next show that every 2 by 2 matrix with determinant 1 has an associated family of conics. If M is a 2 by 2 matrix with trace T and determinant 1, we will show that there is a rotation with matrix R and and a matrix $\overline{M}$ aligned with the coordinate axes, that is has the form $\overline{M} = \begin{bmatrix} \frac{T}{2} & a \\ \frac{T^2-4}{4a} & \frac{T}{2} \end{bmatrix}$, so that $M = R^{-1}\overline{M}R$. Geometrically we are rotating the plane carrying out a transformation that is aligned with the coordinate axes and then rotate the plane back. It follows that the action of M is aligned with the image of the coordinate axes under R^{-1}.

LEMMA 39. *Let M be a 2 by 2 matrix with trace T and determinant $\Delta = \pm 1$ that is not a dilation. Then there exists a rotation $r_{0,\theta}$ with matrix R so that $\overline{M} = RMR^{-1} = \begin{bmatrix} \frac{T}{2} & w \\ \frac{T^2-4\Delta}{4w} & \frac{T}{2} \end{bmatrix}$, for some $w \neq 0$. Furthermore, M and $\overline{M}$ have the same eigenvalues. Finally, X is an eigenvector of M if and only if $\overline{X} = RX$ is an eigenvector of $\overline{M}$.*

PROOF. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then $R^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Let $\overline{M} = RMR^{-1} = \begin{bmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{bmatrix}$

By direct computation, $\begin{cases} \overline{a} &=& a\cos^2\theta - (b+c)\sin\theta\cos\theta + d\sin^2\theta \\ \overline{b} &=& (a-d)\sin\theta\cos\theta + b\cos^2\theta - c\sin^2\theta \\ \overline{c} &=& (a-d)\sin\theta\cos\theta - b\sin^2\theta + c\cos^2\theta \\ \overline{d} &=& a\sin^2\theta + (b+c)\sin\theta\cos\theta + d\cos^2\theta \end{cases}$

We eliminate some special case first. Suppose that $a = d$ and $b \neq 0$; then M is already in the required form. Suppose that $a = d$ and $b = c = 0$; then M is a dilation. Finally, suppose that $a = d$,$b = 0$ and $c \neq 0$. In this case, let $\theta = \frac{\pi}{2}$ and then $\overline{M} = RMR^{-1} = \begin{bmatrix} a & -c \\ 0 & a \end{bmatrix}$ as required. Henceforth we assume that $(a-d) \neq 0$. Setting $\overline{a} = \overline{d}$ gives

$(a-d)(\cos^2\theta - \sin^2\theta) - (b+c)2\sin\theta\cos\theta = 0$ or

$$(a-d)(\cos 2\theta) - (b+c)\sin 2\theta = 0 \tag{1}$$

Solving for θ, we have $\cot 2\theta = \frac{b+c}{a-d}$. It remains to show that $\overline{b} \neq 0$. As we saw above, the rotation by $\frac{\pi}{2}$ will convert a matrix with $b = 0$ but $c \neq 0$ into a matrix of the required form. Since such a rotation composes with R to give a single rotation, we need only eliminate the case where both $\overline{b}$ and $\overline{c}$ are 0. But if these are both 0, RMR^{-1}, and hence M, is a dilation. Therefore, $\overline{M}$ has the form $\begin{bmatrix} \frac{\overline{T}}{2} & w \\ \frac{\overline{T}^2-4\overline{\Delta}}{4w} & \frac{\overline{T}}{2} \end{bmatrix}$, for some $w \neq 0$, where $\overline{T}$ and $\overline{\Delta}$ are the trace and determinant of $\overline{M}$. Since R and R^{-1} both have determinant 1, $\overline{M}$ and M have the same determinant and, by direct computation, $\overline{T} = \overline{a} + \overline{d} = a + d = T$. $\square$

EXERCISE 7.16. *Workout the following examples:*

(i) $M = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

(ii) $M = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$

(iii) $M = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix}$

(iv) $M = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

LEMMA 40. *Let M be any nonsingular matrix with trace T and determinant Δ. Let $\delta = \sqrt{|\Delta|}$ and let D be the dilation $\begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix}$. Then $M = D\overline{M}$ where $\overline{M}$ has trace $\overline{T} = \frac{1}{\delta}T$ and determinant $\overline{\Delta} = \frac{\Delta}{|\Delta|} = \pm 1$. Furthermore, λ is an eigenvalue for M if and only if $\frac{\lambda}{\delta}$ is an eigenvalue for $\overline{M}$ and X is an eigenvector for M with eigenvalue λ if and only if X is an eigenvector for $\overline{M}$ with eigenvalue $\frac{\lambda}{\delta}$.*

PROOF. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let $\overline{M} = \overline{D}M$, where $\overline{D} = \begin{bmatrix} \frac{1}{\delta} & 0 \\ 0 & \frac{1}{\delta} \end{bmatrix}$. Clearly, $\overline{M} = \begin{bmatrix} \frac{a}{\delta} & \frac{b}{\delta} \\ \frac{c}{\delta} & \frac{d}{\delta} \end{bmatrix}$. So $M = (D\overline{D})M = D\overline{M}$. It is also clear that $\overline{T} = \frac{1}{\delta}T$ and $\overline{\Delta} = \frac{1}{\delta^2}\Delta = \frac{\Delta}{|\Delta|} = \pm 1$. Let λ be an eigenvalue for M with eigenvector X: $MX = \lambda X$. Multiply both sides of this equation by $\overline{D}$ on the left to get $\overline{M}X = \lambda \overline{D}X = \lambda \frac{1}{\delta}X = \frac{\lambda}{\delta}X$. So X is also an eigenvector for $\overline{M}$ with eigenvalue $\frac{\lambda}{\delta}$. Now suppose that $\frac{\lambda}{\delta}$ is an eigenvalue for $\overline{M}$ with eigenvector X: $\overline{M}X = \frac{\lambda}{\delta}X$. Multiply both sides of this equation by D on the left to get $MA = D\overline{D}MX = \frac{\lambda}{\delta}DX = \frac{\lambda}{\delta}\delta X = \lambda X$. So X is also an eigenvector for M with eigenvalue λ. $\square$

As we pointed out at the beginning of this section, if $f(X) = MX + B$ is an affinity with a fixed point C then the action of f on the plane about the center C is the same as the action of $g(X) = MX$ about the origin. Using the above three lemmas we understand such action completely. What we must still explor are the affinities that do not have a fixed point.

Consider the affinity $f(X) = MX + B$. Recall that f has a unique fixed point if and only if the matrix $M - I$ has an inverse. So f will not have a unique fixed point if $M - I$ is singular, that is if $\Delta(M - I) = 0$. Our first observation is that the condition that $\Delta(M - I) = 0$ is equivalent to say that 1 is an eigenvalue for M. Since the product of the eigenvalues is $\Delta = \Delta(M)$ the other eigenvalue is Δ and so $1 + \Delta = T$, the trace of M. The next observation is that there are just two fundamentally different ways to avoid having a unique fixed point: no fixed point or more than one fixed point. If f fixes two points then it fixes every point on the line through those points, for example a reflection. If it fixes three non-colinear points then it must be the identity map. If it has no fixed points it still may have a line that is fixed (mapped onto itself), for example a glide reflection. Finally there may be no points and no lines fixed. We will consider each of the following five cases separately:

Case 1, f is the identity map;
Case 2, f is a matrix linear function with one line of fixed points;
Case 3, f is a matrix linear function with one fixed line;
Case 4, f is a matrix linear function with more than one fixed line;
Case 5, f is a matrix linear function with no line of fixed points and no fixed lines.

Let $f(X) = MX + B$, where $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} u \\ v \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$
We will explore the constraints on the unknowns a, b, c, d, u and v in each of the above cases. We already have one constrain that holds in all cases: $a + d = 1 + \Delta$.

EXERCISE 7.17. *Let* $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *where* $\Delta = \Delta(M)$ *and the trace* $T = 1 + \Delta$. *Prove the following.*

(i) M *has eigenvalues 1 and* Δ.
(ii) $bc = (1 - a)(a - \Delta)$
(iii) *If* $bc \neq 0$, *then* M *can be rewritten in the form*
$$M = \begin{bmatrix} a & \frac{1-a}{w} \\ w(a - \Delta) & 1 - (a - \Delta) \end{bmatrix}, \text{ where } w \neq 0 \text{ and } a \neq 1, \Delta.$$
(iv) *If* $bc = 0$, *then* M *is one of the following eight matrices*
$$\begin{bmatrix} 1 & 0 \\ c & \Delta \end{bmatrix}, \begin{bmatrix} \Delta & 0 \\ c & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}, \begin{bmatrix} 1 & b \\ 0 & \Delta \end{bmatrix}, \begin{bmatrix} \Delta & b \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix},$$
$$\begin{bmatrix} 1 & 0 \\ 0 & \Delta \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ where } b \neq 0,\ c \neq 0 \text{ and } \Delta \neq 1.$$

Returning to our investigation of the five cases. We note that, if M is the identity matrix and B the zero vector, then f is the identity map. It is not hard to see that the converse is also true:

EXERCISE 7.18. *Prove that if* f *is the identity map, then* M *is the identity matrix and* B *the zero vector.*

Assume that $f(X) = MX + B$. How can we see if f has a line of fixed points or a fixed line? Suppose $y = mx + p$ is a line of fixed points. Then

$$M \begin{bmatrix} x \\ mx + p \end{bmatrix} + B = \begin{bmatrix} x \\ mx + p \end{bmatrix}, \text{ for all } x.$$

If f only fixes the line, then the restriction of f to the line must be a linear transformation of line that fixes no point, i.e. a translation of the line. In this case

$$M \begin{bmatrix} x \\ mx + p \end{bmatrix} + B = \begin{bmatrix} x + t \\ m(x+t) + p \end{bmatrix}, \text{ for some fixed } t \text{ and all } x.$$

Hence, we can consider the case of a line of fixed points as the special case of a fixed line with $t = 0$.

Assume then that $f(X) = MX + B$ has no fixed point, that $M = \begin{bmatrix} a & \frac{1-a}{w} \\ w(a-\Delta) & 1-(a-\Delta) \end{bmatrix}$, where $w \neq 0$ and $a \neq 1, \Delta$, $B = \begin{bmatrix} u \\ v \end{bmatrix}$ and that f fixes the line $y = mx + p$. We have the equation:

$$\begin{bmatrix} a & \frac{1-a}{w} \\ w(a-\Delta) & 1-(a-\Delta) \end{bmatrix} \begin{bmatrix} x \\ mx + p \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x + t \\ m(x+t) + p \end{bmatrix},$$

for some fixed t and all x. Equating the top components of both sides gives

$$ax + \frac{1-a}{w}(mx + p) + u = x + t.$$

This equation must hold for all x; so, the coefficient of x $(1-a)(\frac{m}{w} - 1)$ must equal 0. Since $a \neq 1$ we must have $m = w$ Then the top component equation becomes $\frac{1-a}{w}p + u = t$. Substituting w for m, the bottom component equation becomes:

$$w(a-\Delta)x + (1 + \Delta - a)(wx + p) + v = wx + wt + p.$$

When like terms are collected, the terms involving x cancel out and we have $(\Delta - a)p + v = wt$. Now eliminating t from $\frac{1-a}{w}p + u = t$ and $(\Delta - a)p + v = wt$ yields $v = wu + (1 - \Delta)p$. Hence M fixes the line $y = wx + \frac{v - uw}{1-\Delta}$ and the action of f on this line is the translation by $t = \frac{\Delta - a}{1-a}u + \frac{1}{w}v$.

So far in this discussion, we have neglected to check for a fixed vertical line, a line with equation $x = q$. We do that now:

$$\begin{bmatrix} a & \frac{1-a}{w} \\ w(a-\Delta) & 1-(a-\Delta) \end{bmatrix} \begin{bmatrix} q \\ y \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} q \\ y + t \end{bmatrix},$$

for some fixed q and t and all y. Equating the top components of both sides gives

$$aq + \frac{1-a}{w}y + u = q.$$

This equation must hold for all y; so, the coefficient of y $\frac{1-a}{w}$ must equal 0. We conclude that fixed vertical lines are possible only in some of the special cases listed above.

We can draw the following conclusions:

LEMMA 41. *Consider the matrix linear equation*

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & \frac{1-a}{w} \\ w(a-\Delta) & 1-(a-\Delta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix},$$

where $w \neq 0$ *and* $a \neq 1, \Delta$.

If $\Delta \neq 1$, f *has a unique fixed line* $y = wx + p$ *where* $p = \frac{v-wu}{1-\Delta}$. *Furthermore, it is point wise fixed if and only if* $v = \frac{\Delta-a}{1-a}wu$.

If $\Delta = 1$ *and* $v = wu$, f *has a family of parallel fixed lines* $y = wx+p$ *for all* p. *Furthermore, the line with* $p = \frac{v}{a-\Delta}$ *is point wise fixed.*

If $\Delta = 1$ *and* $v \neq wu$, f *has no fixed lines.*

Consider $f(X) = MX + B$, where $M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. Here we have $\Delta(M) = 3$ and $T = 2$. Hence the characteristic polynomial is $\lambda^2 - 4\lambda + 3$, which factors $(\lambda-1)(\lambda-3)$. Hence 1 and Δ are the eigenvalues of M and the above lemma applies. We have $\Delta = 3$, $a = 2$, $w = -1$ and $p = 1$. So $y = -x+1$ is a fixed line and, since $v = \frac{1}{-1}(-1)u$, it is fixed point wise:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ -x+1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ -x+1 \end{bmatrix},$$

is easily checked. The top coordinates are $2x + 1(-x+1) - 1 = x$ and the bottom coordinates are $1x + 2(-x+1) - 1 = -x+1$.

Changing B to $\begin{bmatrix} -3 \\ -1 \end{bmatrix}$ gives $p = 2$ and $-x+2$ as a fixed line that is no longer fixed point wise ($v \neq \frac{1}{-1}(-1)u$):

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ -x+2 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} =?$$

The top coordinates are $2x + 1(-x+2) - 3 = x-1$. So the translation along this line must be by $t = -1$. Computing the bottom coordinates, we see that this is so $(x + 2(-x+2) - 1 = -x+3 = -(x-1)+2)$, giving

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ -x+1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} (x-1) \\ -(x-1)+2 \end{bmatrix}$$

Exercise 7.19. *Workout the action of $f(X) = MX + B$ in each of the following cases:*

(i) $M = \begin{bmatrix} -1 & 1 \\ -3 & \frac{5}{2} \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$;
(ii) $M = \begin{bmatrix} -1 & 1 \\ -3 & \frac{5}{2} \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$;
(iii) $M = \begin{bmatrix} \frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{3}{2} \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$;
(iv) $M = \begin{bmatrix} \frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{3}{2} \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$;

We now turn to the special cases in Lemma **??** (iv). Consider the first case, $M = \begin{bmatrix} 1 & 0 \\ c & \Delta \end{bmatrix}$, where $c \neq 0$ and $\Delta \neq 0, 1$. Suppose that $y = mx + p$ is a fixed line:

$$\begin{bmatrix} 1 & 0 \\ c & \Delta \end{bmatrix} \begin{bmatrix} x \\ mx + p \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x + t \\ m(x + t) + p \end{bmatrix}$$

Computing the top coordinate of the left hand side we get $x + u$. So the fixed line is translated by $t = u$ and we must solve

$$\begin{bmatrix} 1 & 0 \\ c & \Delta \end{bmatrix} \begin{bmatrix} x \\ mx + p \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x + u \\ m(x + u) + p \end{bmatrix}$$

for m and p.

Computing the bottom coordinates, we get: $cx + \Delta mx + \Delta p + v = m(x + u) + p$. The coefficient of x must be zero so we have the two equations: $c + m\Delta - m = 0$ and $p\Delta + v - mu - p = 0$. Solving, we have $m = \frac{c}{(1-\Delta)}$ and $p = \frac{v}{(1-\Delta)} - \frac{cu}{(1-\Delta)^2}$. Let's consider an example:

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

We see by inspection that $\Delta = -1$, $c = 1$, $u = -1$ and $v = 1$; we conclude that $m = \frac{1}{2}$ and $p = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. Checking, we do have

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ \frac{1}{2}x + \frac{3}{4} \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x - 1 \\ \frac{1}{2}(x - 1) + \frac{3}{4} \end{bmatrix}.$$

We can make several observations. First $y = mx + p$ will be a line of fixed points if and only if $u = 0$. If both u and v are zero, the origin is fixed and $y = mx$ is the line of eigenvectors for the eigenvalue 1; in this case, the y-axis is the line of eigenvectors for the eigenvalue Δ. If either u or v is not zero then there are no vertical fixed lines - this is easy to check.

Moving on to a matrix linear function based on the second matrix in Lemma **??** (iv)

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \begin{bmatrix} \Delta & 0 \\ c & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix},$$

we easily see that no line of the form $y = mx + p$ is fixed. It is also easy to see that the vertical line $x = \frac{u}{1-\Delta}$ is a fixed line with its points translated by $t = \frac{cu}{1-\Delta} + v$. This will be a line of fixed points when $t = 0$, i.e. when $v = \frac{c}{\Delta-1}u$.

The third matrix in the list is even easier to analyze: it has no fixed lines of the form $y = mx + p$ and it has fixed vertical lines only if $u = 0$. If $u = 0$ then each vertical line $x = q$ is fixed and the translation constant $t = cq + v$.

EXERCISE 7.20. *Compute the fixed lines for the matrix linear functions based on the remaining five matrices listed in Lemma* **??** *(iv).*

7.5. Markov Processes

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