Mathematical Social Sciences 64 (2012) 263–265

Contents lists available at SciVerse ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/econbase

There are more strategy-proof procedures than you think

Donald E. Campbell [a], Jack Graver [b], Jerry S. Kelly [c,*]

[a] Department of Economics and the Program in Public Policy, The College of William and Mary, PO Box 8795, Williamsburg, VA 23187-8795, United States
[b] Department of Mathematics, Syracuse University, Syracuse NY 13244, United States
[c] Department of Economics, Maxwell Hall, Syracuse University, Syracuse, NY 13244-1090, United States

ARTICLE INFO

Article history:
Received 2 June 2011
Received in revised form
30 April 2012
Accepted 3 May 2012
Available online 23 May 2012

ABSTRACT

With as few as eight individuals and five alternatives, there are 561, 304, 372, 286, 875, 579, 077, 983 strategy-proof social choice rules.

The Gibbard–Satterthwaite theorem, now almost 30 years old, has become a staple of microeconomic theory, social choice theory, and mechanism design (Gibbard, 1973; Satterthwaite, 1975). Everyone understands that there are very few strategy-proof procedures: As many dictatorial rules as there are individuals and a handful of rules that have small range, that select less than all alternatives. But there are more strategy-proof rules than you think.

Suppose that there are m alternatives, making up the set X, and n individuals, making up the set N. A *coalition* is a subset of N. For simplicity, assume that preferences are strong, i.e., antisymmetric as well as transitive. There are then $m!$ possible preference orderings. A *profile* u is an assignment of one preference ordering, $\succ_i$, to each individual $i : u = (\succ_1, \succ_2, \dots, \succ_n)$ and so there are $(m!)^n$ profiles. A social choice rule selects one of the alternatives at each profile. There are, therefore, $m^{(m!)^n}$ social choice rules. *But* how many of these rules are strategy-proof?

Classified by the size of their range, there are three kinds of strategy-proof rules.

(1) Rules with $|\text{Range}(f)| = 1$. There are m such constant rules.
(2) Rules with $|\text{Range}(f)| > 2$. Such rules are necessarily dictatorial by the Gibbard–Satterthwaite theorem. So for each such range there are n rules, one for each possible dictator. Of the $2^m - 1$ non-empty subsets of X, i.e., possible ranges, m are singletons and $\binom{m}{2} = \frac{m(m-1)}{2}$ are pairs, so there are $2^m - m - \frac{m(m-1)}{2} - 1$ possible ranges of three or more alternatives. Altogether then, there are $n[2^m - m - \frac{m(m-1)}{2} - 1]$ strategy-proof rules in this category.
(3) Rules with $|\text{Range}(f)| = 2$. There are, as noted earlier, $\frac{m(m-1)}{2}$ possible ranges of two alternatives. Suppose that the range of rule f is $\{x, y\}$. If f is strategy-proof, then the value of f at profile u is entirely determined by how individuals order just x and y at u. If any other part of an individual's ordering affects the social choice, that individual could manipulate the social choice rule. Accordingly, f is completely characterized by the collection $\mathbb{C}$ of *winning coalitions* for x against y, those coalitions at which $f(u) = x$ when x is preferred to y by exactly the members of that coalition i.e.,

$$f(u) = x \quad \text{if and only if} \quad \{i/x \succ_i y\} \in \mathbb{C}.$$

The best known example, for an odd number of individuals, is simple majority voting between a pair of alternatives. In this strategy-proof case, $\mathbb{C}$ is the collection of all coalitions of cardinality greater than $n/2$. It is important to observe that this collection satisfies the superset property:

$$J \in \mathbb{C} \quad \text{and} \quad J \subseteq J^* \subseteq N \text{ implies } J^* \in \mathbb{C}.$$

This generalizes: A rule with range $\{x, y\}$ is strategy-proof if and only if the collection $\mathbb{C}$ of coalitions winning for x against y satisfies the above superset property. Now suppose that we are given a collection of coalitions satisfying the superset property, and define rule f by the condition

$$f(u) = x \quad \text{if and only if} \quad \{i/x \succ_i y\} \in \mathbb{C}.$$

Then f is certainly strategy-proof, but does not necessarily have range $\{x, y\}$. For example, if $\mathbb{C} = 2^N$, then the f determined by $\mathbb{C}$ would be a constant rule always selecting x. At the other extreme, if $\mathbb{C}$ is empty, so there is no coalition winning for x, the f determined by $\mathbb{C}$ would be a constant rule always selecting y. By a *chain* we mean a collection of coalitions, $\mathbb{C}$, that satisfies the superset property. For all chains except the empty collection or $\mathbb{C} = 2^N$, the

* Corresponding author.
E-mail address: jskelly@syr.edu (J.S. Kelly).

0165-4896/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.mathsocsci.2012.05.001

rule defined by $f(u) = x$ if and only if $\{i/x \succ_i y\} \in \mathbb{C}$, is a strategy-proof rule with a range of two. Counting the strategy-proof rules with range of two is equivalent to counting chains.

Because a chain satisfies the superset property, it is entirely characterized by its minimal coalitions, where C is a minimal coalition of $\mathbb{C}$ if (1) C is an element of $\mathbb{C}$ but (2) no proper subset of C is an element of $\mathbb{C}$. The collection M, of minimal coalitions for a strategy-proof rule satisfies the property:

$$\text{If } A, B \in M \text{ and } A \neq B, \text{ then neither } A \subseteq B \text{ nor } B \subseteq A. \qquad (*)$$

(In addition to serving as a determinant of strategy-proof rules, minimal coalitions, according to Riker's size principle (Riker, 1962), are the winning coalitions we expect to actually observe as they evolve from political competition.)

Any collection M of subsets of N satisfying condition $(*)$ is called an *antichain* of N. And any collection $\mathbb{C}$ of subsets of N satisfying the superset property is called a *chain* of N. But just as there are chains of N that do not determine a strategy-proof rule of range two, namely the empty collection and 2^N so there are two antichains that do not determine strategy-proof rules of range two, namely the empty collection, which yields the constant rule always selecting y, and $\{\emptyset\}$, which yields the rule always selecting x.

Summarizing, then, the number of strategy-proof rules on two alternatives with a range of both alternatives is $M(n) - 2$ where $M(n)$ is the number of antichains on N. To illustrate, if $n = 3$, the list of 20 antichains of $\{1, 2, 3\}$ is

Antichains

$\{1, 2, 3\}$
$\{1, 2\}, \{1, 3\}, \{2, 3\}$
$\{1, 2\}, \{1, 3\}$
$\{1, 2\}, \{2, 3\}$
$\{1, 3\}, \{2, 3\}$
$\{1, 2\}$
$\{1, 3\}$
$\{2, 3\}$
$\{1\}, \{2, 3\}$
$\{2\}, \{1, 3\}$
$\{3\}, \{1, 2\}$
$\{1\}, \{2\}, \{3\}$
$\{1\}, \{2\}$
$\{1\}, \{3\}$
$\{2\}, \{3\}$
$\{1\}$
$\{2\}$
$\{3\}$
$\{\emptyset\}$ *
$\emptyset$ *

The number, $M(n)$, of antichains on $\{1, 2, \ldots, n\}$ is the *Dedekind number for n* and $M(3) = 20$. Because we want to exclude the two collections (marked by an asterisk) inducing a range of just one alternative, there are $M(n) - 2$ antichains corresponding to rules f with $|\text{Range}(f)| = 2$ and so just that many strategy-proof rules on two alternatives. Altogether, then, there are $\frac{m(m-1)}{2}[M(n) - 2]$ strategy-proof rules with $|\text{Range}(f)| = 2$. (The reader might object to including the rule with singleton minimal coalition, $\{1\}$, because that means #1 is a dictator and we seem to have already included the dictatorial rules. But that is not correct. There is a dictatorial rule for each possible range of more than one alternative. We had already counted the dictatorial rules that have $|\text{Range}(f)| > 2$ and now we are adding in the dictatorial rules with $|\text{Range}(f)| = 2$.)

We now illustrate this further, and exploit the astonishing rapidity with which $M(n)$ grows with n. Suppose that $m = 5$ and $n = 8$, the largest n for which there is an exact calculation of $M(n)$ (Wiedemann, 1991) (The known exact values of $M(n)$ are available at the Online Encyclopedia of Integer Sequences and the link is in the Reference list. Wiedemann reports his calculation took about 200 h on a Cray-2 processor.) There are 5 constant strategy-proof rules with $|\text{Range}(f)| = 1$. There are 16 possible ranges of 3 or more alternatives and so $16 \times 8 = 128$ dictatorial strategy-proof rules with $|\text{Range}(f)| > 2$. The number of strategy-proof rules with $|\text{Range}(f)| = 2$ is $10 \times [M(8) - 2]$ or $10 \times 56, 130, 437, 228, 687, 557, 907, 785 = 561, 304, 372, 286, 875, 579, 077, 850$. Altogether, there are 561, 304, 372, 286, 875, 579, 077, 983 strategy-proof rules.

Of course if we impose criteria in addition to strategy-proofness, trying to make rules more acceptable, we will reduce this number considerably, as we explore next.

(I) (Anonymity) Suppose, for example, that we wish to impose anonymity, i.e., make individuals interchangeable. That will permit the m constant rules but exclude all the dictatorial rules with $|\text{Range}(f)| > 2$. For strategy-proof rules f with $|\text{Range}(f)| = 2$, these must be rules where the minimal winning coalitions are all the same size. But the rules where the minimal coalitions are all the same size include much more than just the anonymous rules. For the anonymous rules, the minimal coalitions are exactly *all* the coalitions of size t for some $n > t > 1$. (Of the winning coalitions for x against y, pick one of smallest size. Any other coalition of that size can be obtained by a permutation of individuals and, by anonymity, must be winning for x against y. Any other winning coalition must contain one of these as a subset.) Thus there are $n-2$ anonymous, strategy-proof rules f with $|\text{Range}(f)| = 2$.

(II) (Neutrality) Now suppose that we wish to allow violations of anonymity, but impose neutrality, i.e., make alternatives interchangeable. What about the neutral, strategy-proof rules f with $|\text{Range}(f)| = 2$? That will allow all the dictatorial rules with $|\text{Range}(f)| > 2$, but exclude all the m constant rules. What remains is to count the neutral strategy-proof rules f that have $|\text{Range}(f)| = 2$. (Of course the range condition is assured by neutrality.)

This is a larger class and includes all weighted majority rules (with non-negative weights) that are also resolute (i.e., preclude ties). Thus the count of decisive weighted majority voting rules will provide a lower bound on the number of neutral strategy-proof rules with $|\text{Range}(f)| = 2$. For a study of minimal winning coalitions for weighted majority voting rules, see Fishburn and Brams (1996) as well as Nitzan and Paroush (1981). But there do exist neutral strategy-proof rules that are not weighted majority rules; see Campbell and Kelly (2010). An analysis of the antichains that yield neutral, decisive, strategy-proof rules is found in Campbell and Kelly (2012), where for example, it is shown that for $n = 5$, there are 611 neutral, strategy-proof rules f with $|\text{Range}(f)| = 2$. For alternative analyses of the structure of neutral strategy-proof rules, see Jain (1988) as well as a characterization based on ultrafilters in Blau and Brown (1989).

The number $\Psi(n)$ of neutral strategy-proof rules f that have $|\text{Range}(f)| = 2$ grows at least exponentially:

Proposition. $\Psi(n) \geq 2^{n-1}$.

Proof. Of the 2^n coalitions, half, or 2^{n-1}, are of odd cardinality. The collection of rules that are simple majority voting on these coalitions are all distinct. □

(III) (Anonymity and neutrality) If we impose both anonymity and neutrality then for even n, there are no strategy-proof rules with $|\text{Range}(f)| = 2$, while for odd n, only simple majority voting is strategy-proof with $|\text{Range}(f)| = 2$.

To get a sense of relative magnitudes, if $n = 5$ and $m = 2$, there are $M(5) - 2 = 7578$ strategy-proof rules with $|\text{Range}(f)| = 2$, of which 1 is neutral and anonymous, 3 are anonymous, and 81 are neutral (as determined by Campbell and Kelly (2010)).

(IV) (Full range) Now suppose we allow violations of anonymity and neutrality but want $\text{Range}(f) = X$. With our assumption that preferences are strong, this only allows n (completely) dictatorial rules. But, if we now allow for weak preferences, i.e., non-trivial indifference classes, there is a very large number of dictatorial rules. All that is required for a rule to be dictatorial with dictator #1 is that $f(u)$ is an element of 1's topmost indifference set. But if that topmost set is not a singleton, the choice can be made dependent on others' preferences. One frequently mentioned such rule is serial dictatorship:

(1) If #1's topmost indifference set is a singleton, $f(u)$ is that element;
(2) If #1's topmost indifference set I_1 has more than one element, look at #2's ordering restricted to I_1; if that restricted ordering has a topmost alternative, $f(u)$ is that element;
(3) If #2's topmost indifference set I_2 in the restricted ordering has more than one element, look at #3's ordering restricted to I_2; if that restricted ordering has a topmost alternative, $f(u)$ is that element; and so on.

This construction is driven by the following idea: If I_1 contains three or more alternatives and if f on the domain restricted to having #1's ordering fixed at its value at u has a range of more than three elements, Arrow's theorem applies again and requires a dictator on I_1. But not only might I_1 contain only two alternatives, but even when it contains more than two, the range of f might be only two alternatives for fixed ordering for #1. This observation allows us to find an enormous lower bound for the number of strategy-proof rules even when we require that f have full range.

Consider the following category of rules. Individual #1 is a dictator. But there is a function that maps each subset S of more than one alternative to a pair of alternatives $\{x, y\}$ in S. Then, when $I_1 = S$, one of the $M(n-1)-2$ strategy-proof rules with range $\{x, y\}$ is applied, using the profile restricted to individuals $\{2, 3, \ldots, n\}$. Since there are $2^m - m - 1$ subsets of more than one alternative, there are at least

$$[M(n-1) - 2]^{2^m-m-1}$$

strategy-proof rules. For $m = 5$, and $n = 9$, this is

$$(56, 130, 437, 228, 687, 557, 907, 785)^{26} \approx 10^{591}.$$

There is another interesting connection between the Dedekind numbers and strategy-proof social choice. Although there are not explicit formulas for $M(n)$ or even exact calculations for $m > 8$, there are a variety of established estimates, e.g.,

$$M(n) > 2^{\binom{n}{\lfloor n/2 \rfloor}}$$

where the right-hand side counts the antichains made up of coalitions of size $\lfloor \frac{n}{2} \rfloor$, the greatest integer not exceeding $\frac{n}{2}$. This inequality already illustrates the rapid increase in the number of strategy-proof rules with n. Korshunov (1981) has established more accurate estimates. Zaguia (1993) observes that the main idea behind Korshunov's estimates is that almost all antichains are contained in the union of the collections of the middle three sizes of subsets of X. Thus for most strategy-proof rules, most of the minimal winning coalitions are close to the minimal winning coalitions for simple majority voting.

References

Blau, Julian H., Brown, Donald J., 1989. the structure of neutral monotonic social functions. Social Choice and Welfare 6 (1), 51–61.

Campbell, Donald E., Kelly, Jerry S., 2010. Strategy-proofness and weighted voting. Mathematical Social Sciences 60 (1), 15–23.

Campbell, Donald E., Kelly, Jerry S., 2012. Neutral, Resolute, Strategy-proof social choice rules, Unpublished.

Fishburn, Peter C., Brams, Steven J., 1996. Minimal winning coalitions in weighted-majority voting games. Social Choice and Welfare 13 (4), 397–417.

Gibbard, Allan, 1973. Manipulation of voting schemes: a general result. Econometrica 41 (4), 587–601.

Jain, Satish K., 1988. Characterization of monotonicity and neutrality for binary paretian social decision rules. Mathematical Social Sciences 15, 307–312.

Korshunov, A.D., 1981. The number of monotone Boolean functions. Problemy Kibernetiki 38, 5–108.

Nitzan, Shmuel I., Paroush, Jacob, 1981. The characterization of decisive weighted majority rules. Economics Letters 7, 119–124.

Riker, William H., 1962. Theory of Political Coalitions. Yale, New Haven.

Satterthwaite, Mark, 1975. Strategyproofness and arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. Journal of Economic Theory 10 (2), 187–217.

Sloane, N.J.A., Sequence A014466 in The On-Line Encyclopedia of Integer Sequences. http://www.research.att.com/njas/sequences/A000372.

Wiedemann, Doug, 1991. A computation of the eighth Dedekind number. Order 8 (1), 5–6.

Zaguia, Nejib, 1993. Isotone maps: enumeration and structure. In: Sauer, N.W., Woodrow, R.E., Sands, B. (Eds.), Finite and Infinite Combinatorics in Sets and Logic. Kluwer Academic Publishers, Dordrecht, Boston, pp. 421–430.