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WHEN DOES A CURVE BOUND A DISTORTED DISK?*

JACK E. GRAVER† AND GERALD T. CARGO†

Abstract. Consider a closed curve in the plane that does not intersect itself; by the Jordan–Schoenflies theorem, it bounds a distorted disk. Now consider a closed curve that intersects itself, perhaps several times. Is it the boundary of a distorted disk that overlaps itself? If it is, is that distorted disk essentially unique? In this paper, we develop techniques for answering both of these questions for any given closed curve in the plane.

Key words. immersion, graphite fragments

AMS subject classifications. 05C10, 57M15, 57R42, 58D10

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1. Introduction. The problem of when the boundary of a distorted disk uniquely determines that disk has been considered in several contexts. As a combinatorial problem, it arose naturally in an investigation of graphite fragments by Guo, Hansen, and Zheng (GHZ) [6]. Graphite is one of the crystalline forms of carbon. The carbon atoms form in hexagonal rings, which are attached to one another along edges in such a way as to form "regions," each of which can be pictured as a connected, finite union of closed hexagons in the hexagonal tessellation of the plane. We call such a "region" a *simple graphite fragment* if its boundary is a simple closed curve, as in Figure 1. One can describe a counterclockwise tour of the boundary of a simple graphite fragment by a cyclic sequence of right (R) and left (L) turns. Thus, the sequence of R's and L's (the *boundary code*) uniquely determines the boundary of the fragment and the fragment itself, up to a congruence of the tessellation.

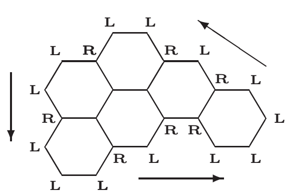

FIG. 1.

To visualize a general *graphite fragment* think of constructing it in space by gluing together hexagonal tiles edgewise. As the fragment is being constructed, it may eventually turn around and build underneath itself. Such a fragment projects onto a region of the hexagonal tessellation of the plane that overlaps itself. The boundary sequence uniquely determines the boundary of this overlapping region, up to a congruence of the tessellation. But, does it still uniquely determine the fragment itself? In their paper, GHZ answered this question in the negative. They produced the simplest example of two nonisomorphic graphite fragments with the same boundary code. We sketch these fragments side by side in Figure 2. When projected onto the hexagonal tessellation of the plane, these fragments are self-overlapping and hard to interpret;

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http://www.siam.org/journals/sidma/25-1/76771.html
†Department of Mathematics, Syracuse University, Syracuse, NY 13244-1150 (jegraver@syr.edu, gtcargo@syr.edu).

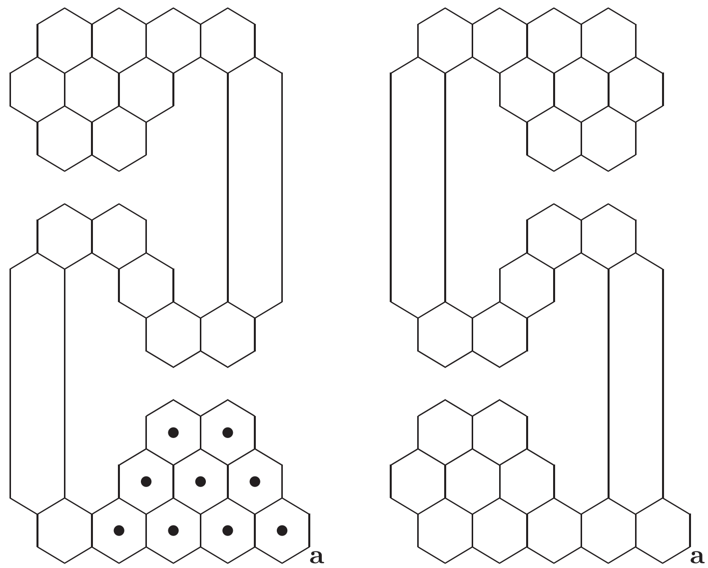

Fig. 2.

hence, we have elongated some of the hexagons to eliminate the overlappings. Starting with a left turn at the vertex labeled a and moving counterclockwise around the fragment, one easily sees that, in both cases, the boundary code is: LLRL RLLR LLRL RRRR RRLR LLRL LRLL RLLR LRLR LLRL LRLL RLLR LRRR RRRL RLLR LLRL LRLL RLRL RLRL. It is easy to see that the two fragments are not isomorphic: the configuration in the left-hand fragment consisting of the 9 hexagons indicated by the dots has no counterpart in the right hand fragment.

Brinkmann, Friedrichs, and von Nathusius [3] generalized the graphic fragment problem to what they called (m, k)-patches $(m, k \geq 3)$. These are fragments constructed by gluing regular m-gon tiles together with exactly k tiles meeting at each interior vertex and at most $k - 1$ tiles meeting at any boundary vertex. Each (m, k)-patch has a natural projection into the (m, k)-tessellation of the euclidean plane, hyperbolic plane, or sphere. Graver [5] then proved that if an (m, k)-patch is not uniquely determined by its boundary, then some polygon is covered three or more times by the projection of the patch onto the corresponding (m, k)-tessellation.

So far we have considered the following question: *Given a fragment, is there another nonisomorphic fragment with the same boundary code?* We could add a question and ask, *Given a boundary code, is there a fragment with that boundary code?* Furthermore, in view of the Brinkmann, Friedrichs, and von Nathusius generalization, it is natural to go one step further and formulate continuous versions of these questions. Actually, these continuous versions have been studied extensively as the simplest case in immersion theory. So that we can appreciate on an intuitive level the continuous versions of the two questions just posed, we shall now proceed very informally and postpone formal, technically correct definitions and statements until the next section.

A continuous function $\phi : S \to T$ from a topological space S into a topological space T is a *local homeomorphism* if, for each point $s \in S$, there exists a neighborhood, N, of s so that the restriction of ϕ to N is a homeomorphism of N onto $\phi(N)$. We may think of an *immersion* as a local homeomorphism with some additional smoothness requirements. Let D denote the open unit disk in $\mathbb{R}^2$, $\overline{D}$ its closure, and $C = \overline{D} - D$ its boundary. We consider two questions.

Question 1. If ζ is an immersion of C into $\mathbb{R}^2$, can ζ be extended to an immersion $\delta : \overline{D} \to \mathbb{R}^2$?

Question 2. If the extension δ requested above exists, is it unique up to an equivalence?

Immersions $\delta : \overline{D} \to \mathbb{R}^2$ and $\delta' : \overline{D} \to \mathbb{R}^2$ are *equivalent* if there exists a homeomorphism $\beta : \overline{D} \to \overline{D}$ such that $\delta \circ \beta = \delta'$.

Of course, an immersion ζ of the circle C can always be extended to a continuous function δ from $\overline{D}$ into $\mathbb{R}^2$. For example, map each horizontal segment of the disk linearly onto the (possibly degenerate) segment joining the images of its endpoints. The condition that the extension δ be an immersion precludes "folding" and "twisting" as pictured in Figure 3 below.

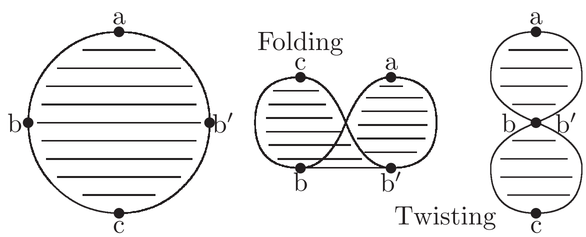

FIG. 3.

In the next section, we trace the early history of the immersion problem. We outline the solutions due to Titus and to Blank in section 3. Both of their solutions to the immersion problem are combinatorial in nature. Rather than give another combinatorial solution to this problem, our approach is to identify an equivalent purely graph-theoretic problem. Specifically, in section 4, we reformulate the questions in terms of graph theory. Section 5 is devoted to a solution of the reformulated, graph-theoretic problem. This approach has several advantages. For example, it enables us to identify several easy-to-check necessary conditions for the existence of an extension. This solution also has the 3-cover theorem as a corollary. Finally, this approach easily extends to the formulation of and a solution to the analogous problem for an immersion of the circle in the sphere (section 6).

2. A short history of the immersion problem. A special case of a slightly modified version of Question 1 was posed by Picard [9] who wanted to extend the Schwarz–Christoffel theorem in conformal mapping theory to the case of nonsimple polygons. Various authors then posed and solved a continuous version of Picard's question.

Before we can discuss these earlier works, we must introduce some additional terminology. A function $\zeta : C \to \mathbb{R}^2$ is a *regular representative* of a closed curve if it has a continuously turning, nonzero tangent vector, $\frac{d}{d\theta}\zeta(e^{i\theta})$. (Some authors refer to a regular representation as an immersion.) A point p in the image of ζ is said to be a *simple crossing point* if p has exactly two preimages and the associated tangent vectors are linearly independent. A regular representative of a closed curve is said to be *normal* if it has a finite number of simple crossing points and every other point in the image has just one preimage. In investigating certain extension questions, Whitney [14] proved that, without loss of generality, attention can be restricted to normal regular representations. Specifically he showed that, given a regular representation ζ, one can obtain by means of an arbitrarily small alteration a normal regular representation ζ^* such that the answers to our two questions are exactly the same for ζ^* and ζ. For example, the complex function $\zeta(z) = z^2$ on C is regular but not normal, while

$\zeta^*(z) = z^2 - \epsilon z$ (for each arbitrarily small positive number ϵ) is normal on C; by Whitney's result answering our questions for ζ^* will give us answers for ζ.

Two regular representations ζ_1 and ζ_2 are *equivalent* if there exists a sense-preserving homeomorphism ϕ of C onto C such that $\zeta_2 = \zeta_1 \circ \phi$ and $\frac{d}{d\theta}\phi(e^{i\theta})$ is continuous and nonzero. A regular curve is defined to be an equivalence class of regular representations. The problems investigated in this paper are of such a nature that any representation can be replaced by one equivalent to it. However, since the notation and language become too cumbersome with equivalence classes, we will usually use representations and replace them when desired.

It is easy to give a heuristic argument that it is reasonable to restrict our attention to normal curves. If two segments of C are mapped by ζ onto the same segment of the image curve, we can alter ζ slightly so that the images of the two segments are distinct or have a simple crossing. Furthermore, this alteration can be carried out simultaneously on all extensions of ζ to the entire disk. Hence the answers to the questions "Is there an extension of ζ to the entire disk, and if so, is it unique?" remain unaltered. We can also use such simultaneous alterations to further simplify the curve by replacing each multiple crossing at a point by several simple crossings, again, without changing the answers to our questions. We illustrate these alterations in Figure 4. After Whitney's paper, all investigators have restricted their investigations to normal regular functions, and so shall we.

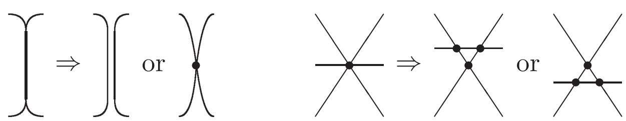

FIG. 4.

If we apply this simplifying process along with smoothing to the boundary of the GHZ example, we get a curve like the one pictured in Figure 5. At this point we should observe that the fact that the two fragments in Figure 2 are not isomorphic as graphs does not immediately imply that the the corresponding continuous extensions of the smoothed boundary code are not homeomorphic. But, as we will eventually prove, they are not homeomorphic. We will use this curve throughout the paper to illustrate the techniques and results.

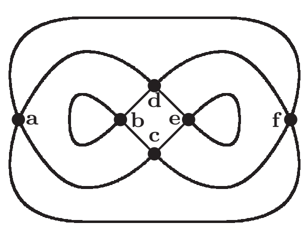

FIG. 5.

Much of immersion theory involves higher-dimensional spaces and is quite abstract. However, initially many important results were obtained in the very simple setting where one considers a normal presentation $\zeta : C \to \mathbb{R}^2$ and asks if ζ has a continuous extension $\delta : \overline{D} \to \mathbb{R}^2$ such that $\delta_{|D}$ is described by a specified member of the following list:

(i) interior mapping with no branch points,
(ii) locally homeomorphic mapping,

(iii) immersion,
(iv) locally diffeomorphic mapping,
(v) analytic (i.e., holomorphic) function with no critical points.

(Some authors replace (i) by "(i′) interior mapping" and study the branch points.)

By definition, a function mapping D into $\mathbb{R}^2$ is interior if it is continuous, open (i.e., maps each open subset of D onto an open subset of $\mathbb{R}^2$), and light (i.e., the preimage of any point is totally disconnected). An immersion is a C^1 mapping that has maximal rank at each point of D; in the present setting, the inverse mapping theorem implies that an immersion is a locally diffeomorphic mapping; that is, conditions (iii) and (iv) are equivalent. Since the square of the modulus of the derivative of a holomorphic function is equal to the Jacobian determinant of the induced transformation, condition (v) entails condition (iv). In summary, each condition entails the previous condition, and conditions (iii) and (iv) are equivalent.

Titus [13] calls a normal representation ζ an *analytic boundary* if it has a continuous extension to $\overline{D}$ whose restriction to D is analytic. He calls a normal representation ζ an *interior boundary* if it has a continuous extension to $\overline{D}$ whose restriction to D is interior and sense-preserving. A nonconstant analytic function $F : D \to \mathbb{R}^2$ and its complex conjugate $\overline{F}$ are both interior. Moreover, F is sense-preserving and $\overline{F}$ is sense-reversing. The composition of an analytic function and a homeomorphism is an interior function. From Stoïlow's theorem [13, p. 45] and the classification theorem in the theory of Riemann surfaces, one can prove that if $F : D \to \mathbb{R}^2$ is sense-preserving, interior, and bounded, then there exists a sense-preserving homeomorphism H of D onto D and an analytic function $W : D \to \mathbb{R}^2$ such that $F = W \circ H$.

About 1948 Loewner [8] suggested the following problem.

Problem I (immersion). Given a normal representation ζ, find necessary and sufficient conditions that ζ be an interior boundary.

In 1961 Titus [13] gave a complete answer, from the point of view of combinatorial topology, to Loewner's problem. In 1967 in his dissertation Blank [2] (cf. 11) gave a complete solution in terms of combinatorial invariants of the related problem for immersions.

As noted above, an immersion is an interior mapping (without branch points). Moreover, Jewett [7, Theorem 3, p 111] proved that if a normal representation ζ has a continuous extension to $\overline{D}$ whose restriction to D is an interior mapping without branch points, then ζ has a continuous extension to $\overline{D}$ whose restriction to D is an immersion. So Blank solved Problem I in the case where branch points are excluded. In summary, Problem LH below is equivalent to Problem I if branch points are excluded.

Problem LH (local homeomorphism). Given a normal representation ζ, find necessary and sufficient conditions where ζ has a continuous extension to $\overline{D}$ whose restriction to D is locally homeomorphic and sense-preserving.

As Titus [13] points out, the theorem stated just before Problem I in conjunction with Carathéodory's extension theorem yields what follows.

THEOREM 1. *Let ζ be an interior boundary having F as its extension to $\overline{D}$. Then there exist a sense-preserving homeomorphism H of $\overline{D}$ onto $\overline{D}$ and a continuous function $W : \overline{D} \to \mathbb{R}^2$ such that $W|_D$ is analytic and $F = W \circ H$.*

This theorem unleashes the power of a strong version of the argument principle when one is dealing with interior boundaries. It also shows that every interior boundary is equivalent to an analytic boundary. In that sense, Problem I is the same as Problem LH except that in the former problem the extension is permitted to have branch points (locally topologically equivalent to the power mapping z^m). In other words, in Problem LH the derivative of the analytic function in question is not permitted to ever equal zero, whereas in Problem I the derivative is permitted to have zeros in D. Local homeomorphisms are easy to construct. Simply let $p(z) = \int_0^z q(\zeta)d\zeta$, where q is any polynomial whose zeros lie outside $\overline{D}$.

3. The results of Titus and Blank. Titus approached the problem by triplicating a segment of the curve between two crossing points or *vertices* and splitting the curve into two simpler curves. In Figure 6, we illustrate the *cut,* as Titus called it, at the segment from **a** through **c** to **e**. Titus then proved that the curve S bounds a distorted disk if and only if each of the two curves S_1 and S_2 of any cut bound a distorted disk. In our example, S_1 clearly bounds a distorted disk, and it is not too hard to see that S_2 does too. Titus also proved that one can always find a cut so that the curves S_1 and S_2 are both simpler (have fewer crossing points) than S. The problem can then be solved inductively. Titus did not consider the uniqueness question.

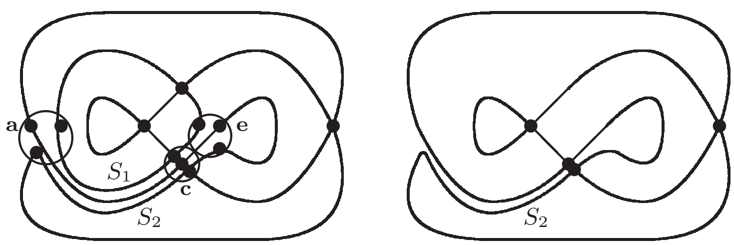

FIG. 6.

Both the existence and the uniqueness problems were considered by Blank in his dissertation [2] (cf. 11). He also used an inductive approach cutting the curve into two or more simpler curves. However, he defined his "cuts" differently. He starts by selecting a point interior to each region and drawing disjoint rays, one from each of these points, satisfying the following two conditions:

(i) The rays do not passes through any of the crossing points of the curve;
(ii) No ray intersects any of the segments between two crossing points in more than one point.

Blank first proves that the original curve can always be redrawn so that this construction can be carried out. In Figure 7, we have drawn such a family of rays for our example.

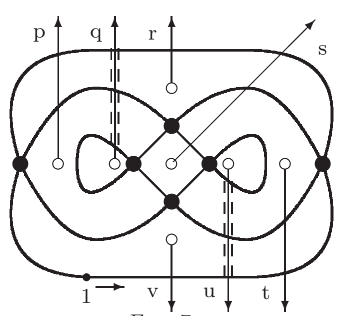

FIG. 7.

Blank labeled these rays with the letters from an alphabet and constructed a word on this alphabet as follows: select a noncrossing point on the curve (for example, the point labeled 1) and then traverse the curve in the counterclockwise direction listing the ray labels as we encounter them; specifically, if the ray crosses the curve from left to right, we simply list its label, and if it crosses from right to left, we list the inverse of its label. We may eliminate the artificial choice of starting point by thinking of this sequence of letters and their inverses as a circular sequence. The resulting circular sequence is called the "Blank word" of the curve and the given family of rays. For our example we have

$$vutsq^{-1}utsrqpu^{-1}sqp.$$

A letter and its inverse correspond to an "acceptable cut" if neither of the two subwords bounded by this pair of symbols contains an unmatched inverse. For example, q^{-1} and the first q yield the acceptable cut

$$q^{-1}utsrq \qquad \text{and} \qquad qpu^{-1}sqpvutsq^{-1}.$$

Continuing, we dissect the second subword to get

$$q^{-1}utsrq, \qquad u^{-1}sqpvu, \qquad \text{and} \qquad utsq^{-1}qpu^{-1}.$$

These three subwords correspond to the dissection of the original curve into three simple (non-self-intersecting) curves as indicated by the dashed lines above in Figure 7 and pictured in Figure 8.

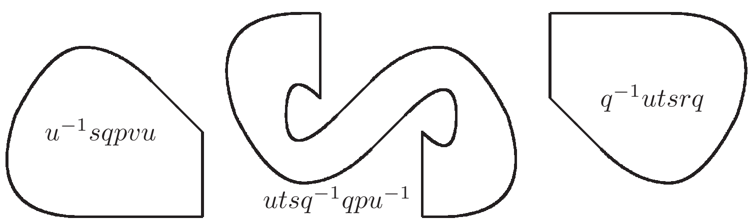

FIG. 8.

Blank then describes an equivalence relation on all decompositions of the initial word corresponding to the equivalence relation on extensions defined above. A second, nonequivalent decomposition is listed below:

$$qpvutsq^{-1}, \qquad utsrqpu^{-1}, \qquad \text{and} \qquad uu^{-1}sqq^{-1}.$$

One of the more difficult things that Blank had to do was to show that his final conclusions were independent of the initial selection of a family of rays.

In [12], Shor and Van Wyk discuss the algorithmic aspects of the Blank approach and then modify that approach developing efficient algorithms for deciding (1) whether or not the curve has an interior, and (2) exactly how many "different" interiors are possible. The also conjectured that if no region is covered more than twice, then all interiors are equivalent. We verify their conjecture in Corollary 3.2.

4. The graph-theoretic formulation. Viewing the GHZ example, Figure 5, we cannot help but observe that the actual shape of the curve is irrelevant as long as the sequence of crossings is preserved. This was formally proved by Titus (Theorem 3 in [13]). Thus, the essential features of our curve are really properties of the embedded plane multigraph that it determines. Our approach to the problem is based on this observation. Hence, we proceed to formulate our questions in this graph theory setting.

Let the normal curve S be the image of the immersion ζ. The counterclockwise orientation of C gives an orientation or direction to the curve S. With this in mind, we define directed plane multigraph $\vec{\Gamma}_\zeta = (V, \vec{E}, F)$ as follows: the vertex set consists of the crossing points of the curve S, the edge set is identified with the directed arcs of S joining these points, and the faces correspond to the regions "cut out" by the curve. In the GHZ example, Figure 9, there are six vertices, $\{a, b, c, d, e, f\}$, eight faces, and 12 edges including multiple edges joining a and f and loops at b and e.

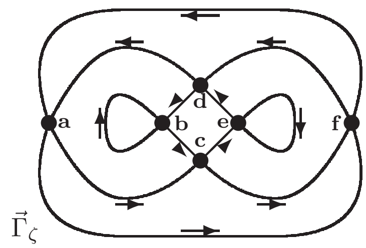

Fig. 9.

Now suppose that ζ may be extended to an immersion δ of the entire closed disk $\overline{D}$. Using our intuitive approach, we visualize $\overline{D}$ as a distorted disk in 3-space with its boundary C projecting down onto S in the plane. For each vertex $v \in V$, let $\{v_0, v_1, \dots\}$ denote the collection of points on $\overline{D}$ that project down onto v, and let $\hat{V}$ denote the collection of all of these points. Intuitively, the preimage of each arc of S consists of one arc on C and perhaps several disjoint copies interior to $\overline{D}$ on other levels. These "lifted" arcs and the "lifted" vertices form a graph drawn on the disk $\overline{D}$. The regions of $\overline{D}$ cut out by this graph plus an outer face define the plane graph $\Phi_\delta = (\hat{V}, \hat{E}, \hat{F})$. We illustrate this lifting construction for the GHZ example in Figure 10, which includes the lifted graphs from two different (nonequivalent) extensions of ζ.

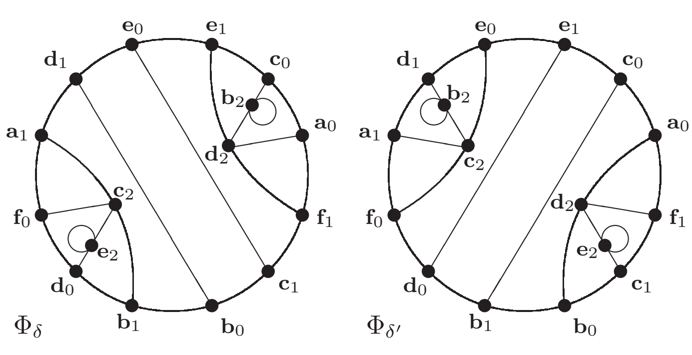

Fig. 10.

Conversely, given an immersion ζ with image S and associated directed multigraph $\vec{\Gamma}_\zeta$ as above and assuming that we can construct a "covering graph" Φ, we could then build and extension δ of ζ by piecing together homeomorphisms between the faces of Φ and the corresponding faces of $\vec{\Gamma}_\zeta$. Intuitively then, the disk immersion problem can be reduced to a graph theory problem. We now wish to put this equivalence on sound footing. We start by developing the graph theory we need to do this.

In this intuitive overview of our plan of attack, we called Φ_δ a "covering graph" of $\vec{\Gamma}_\zeta$. It is, in fact, not a covering graph; the problem comes with the outside face. If Φ_δ were a cover graph of $\vec{\Gamma}_\zeta$, the outside face of Φ_δ would have to be mapped onto some face of $\vec{\Gamma}_\zeta$, and the outside face of $\vec{\Gamma}_\zeta$ would have to be the image of some face of Φ_δ. Neither of these requirements can be met. So while the standard definitions and results from the theory of graph coverings may be used as a guide, we will have to develop a specialized version of this theory to meet our needs. To avoid confusion between standard covering theory and our variation, we shall introduce different terminology adapted from the literature of mathematical chemistry.

A *patch* (a *directed patch*), $\Phi = (V, E, F, \vec{B})$ ($\vec{\Phi} = (V, \vec{E}, F, \vec{B})$), is obtained from a (directed) 2-connected, plane multigraph (loops and multiple edges permitted) (V, E, F') ($(V, \vec{E}, F')$) by deleting the outside face and attaching its directed boundary $\vec{B}$ oriented counterclockwise around the rest of the graph (clockwise around the outside face itself). Requiring a plane multigraph without loops to be 2-connected is equivalent to requiring the boundary of each face to be an elementary circuit. Hence, $\vec{B}$ and the boundaries of the faces of the patch $\Phi = (V, E, F, \vec{B})$ are all elementary circuits when the loops are deleted. The technical problem caused by loops can be observed in Figure 10. We also note at this point that, in a directed patch $\vec{\Phi} = (V, \vec{E}, F, \vec{B})$, the directions assigned to edges in the boundary by $\vec{E}$ and $\vec{B}$ need not agree.

An *immersion* α of the directed graph $\vec{\Delta} = (U, \vec{B})$ onto the directed graph $\vec{\Gamma} = (V, \vec{E})$ is a pair of functions using the same symbol such that $\alpha : U \to V$ is onto, $\alpha : \vec{B} \to \vec{E}$ is a bijection, and all incidences and directions are preserved. By a directed Euler circuit for $\vec{\Gamma} = (V, \vec{E})$, we mean a directed elementary circuit $\vec{\Delta} = (U, \vec{B})$ and an immersion α of $\vec{\Delta}$ onto $\vec{\Gamma}$. If in addition, $\vec{\Gamma}$ is imbedded in the plane, we may choose to require that, in traversing the image of circuit in $\vec{\Gamma}$, one leaves each vertex by the edge opposite from the one along which it entered that vertex. Since a graph that admits an Euler circuit must have only vertices of even degree, the "opposite" edge is well defined. An Euler circuit satisfying this additional requirement is called a *straight-through Euler circuit.* Straight-through paths and circuits were defined and studied by Pisansky, Tucker, and Zitnik [10].

We are now able to define the combinatorial analogue of a normal curve given by a normal immersion of the unit circle. By a *normal patch* we shall mean a directed patch $\vec{\Gamma} = (V, \vec{E}, F, \vec{A})$ such that

(i) each vertex of $\vec{\Gamma}$ has degree 4,

(ii) $\vec{\Gamma}$ admits a straight-through Euler circuit $\vec{\Delta} = (U, \vec{B})$.

Let ζ be a normal immersion of the unit circle C into $\mathbb{R}^2$ with the curve S as its image, directed by the counterclockwise orientation of C. Above, we introduced the directed plane multigraph $\vec{\Gamma}_\zeta = (V, \vec{E}, F)$ with the collection of crossing points of S as its vertex set V, the arcs joining the crossing points as its edge set E, and the connected components of $\mathbb{R}^2 - S$ as its faces. If we redefine F to be all faces except the outside face and include $\vec{A}$, the oriented boundary of the outside face, we have the directed patch $\vec{\Gamma}_\zeta = (V, \vec{E}, F, \vec{A})$. Since ζ is a normal immersion, each vertex of $\vec{\Gamma}_\zeta = (V, \vec{E}, F, \vec{A})$ has degree 4 and is the image of exactly two points on the unit circle C. Let U denote the $2|V|$ preimages of the points in V, and let $\vec{B}$ denote the ordered pairs of consecutive points of U in the counterclockwise orientation of C.

Then $\vec{\Delta} = (U, \vec{B})$ is a straight-through Euler circuit for $\vec{\Gamma}_\zeta$. Hence conditions (i) and (ii) above are satisfied, and $\vec{\Gamma}_\zeta$ is a normal patch—*the normal patch of* ζ.

An *isomorphism* β of a patch $\Phi = (V, E, F, \vec{B})$ onto a patch $\Phi^* = (V^*, E^*, F^*, \vec{B^*})$ is a collection of bijections (all using the same symbol): $\beta : V \to V^*$, $\beta : E \to E^*$, and $\beta : F \to F^*$ such that

(i) all incidences are preserved;

(ii) the restriction of β to $\vec{B}$ is an orientation preserving bijection onto $\vec{B^*}$.

An *immersion* α of a patch $\Phi = (V', E', F', \vec{B})$ onto a normal patch $\vec{\Gamma} = (V, \vec{E}, F, \vec{A})$ is a collection of onto functions (all using the same symbol): $\alpha : V' \to V$, $\alpha : E' \to E$, $\alpha : F' \to F$ such that

(i) all incidences are preserved;

(ii) α restricted to $(U, \vec{B})$, where U is the set of vertices on the boundary, is the straight-through Euler circuit for $\vec{\Gamma}$;

(iii) for each $v' \in V'$, α restricted to v' and the set of edges and faces incident with v' is one-to-one and onto;

(iv) for each $f' \in F'$, α restricted to f' and the set of edges incident with f' is one-to-one and onto.[1]

We say that the Φ is a *covering patch* of the normal patch $\vec{\Gamma}$ if there is an immersion α of Φ onto $\vec{\Gamma}$. The patches in Figure 10 are covering patches for the normal GHZ patch pictured in Figure 5. Two covering patches Φ and Φ^* of $\vec{\Gamma}$ given by immersions α and α^*, respectively, are *equivalent* if there exist a isomorphism $\beta : \Phi \to \Phi^*$ so that $\alpha^* \beta = \alpha$ on the boundary, $\vec{B}$, of Φ. The covering patches in Figure 10 are nonequivalent: there is no patch isomorphism between them that fixes the boundary.

Let $\vec{\Gamma} = (V, \vec{E}, F)$ be any directed multigraph embedded in the plane. By a *flag* in $\vec{\Gamma}$ we mean a triple (v, e, f) consisting of a vertex, edge, and face all incident. If as one moves from v along e (regardless of the orientation of e) f is on the left (right), we say that (v, e, f) is a *left-hand flag* (*right-hand flag*).

Lemma 1.

(i) *Patch isomorphisms and immersions preserve flag orientation and the cyclic orientations of edges around each vertex and face.*

(ii) *If a normal patch* $\vec{\Gamma} = (V, \vec{E}, F, \vec{A})$ *admits a covering patch and if the directions of* $\vec{E}$ *and* $\vec{A}$ *agree on one edge, then they agree on all boundary edges.*

Proof. Let α be an isomorphism of the patch $\Phi = (V', E', F', \vec{B})$ onto the patch $\Phi^* = (V^*, E^*, F^*, \vec{B^*})$ or immersion of $\Phi = (V', E', F', \vec{B})$ onto the normal patch $\vec{\Gamma} = (V, \vec{E}, F, \vec{A})$. Consider a left-hand flag (v', e', f') in Φ with the flag (v, e, f) as its image (under α) in Φ^* or $\vec{\Gamma}$. Since α is one to one and onto on the edges and faces incident with v' and preserves incidences, the counterclockwise ordering of edges and faces around v' is either preserved or reversed by α, as pictured in Figure 11. It follows that all of the flags on v' have their orientations preserved by α or all of the flags on v' have their orientations reversed by α.

Next consider the right-hand flag (w', e', f'), where w' is the other endpoint of e'. Clearly, the orientation of (w', e', f') and all of the flags at w' are preserved (reversed) by α if and only if the orientation of (v', e', f') is preserved (reversed) by α. It follows that, since patches are connected, either the orientations of all flags are preserved by α or the orientations of all flags are reversed by α.

[1]Complete parallelism between conditions (iii) and (iv) is not possible since a nonloop edge in the boundary of a face of Φ may be mapped onto a loop in $\vec{\Gamma}$.

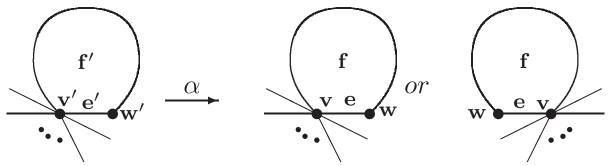

FIG. 11.

Now let $e' = (v', w')$ be a directed edge in $\vec{B}$ that maps onto the directed edge $e = (v, w)$ in $\vec{B^*}$ or onto the directed edge $e = (v, w)$ in $\vec{A}$. Since e' is on the boundary of Φ' and oriented counterclockwise around the patch, it bounds only one face f', and the flag (v', e', f') is a left-hand flag. Similarly, e bounds only one face f, and the flag (v, e, f) is a left-hand flag. But then α must map f' to f and the left-hand flag (v', e', f') to the left-hand flag (v, e, f). Since the orientation of one flag is preserved by α, the orientation of all flags and the cyclic orientation around all vertices are preserved. In particular, each flag (v, e, f) on the boundary $\vec{A}$ is the image of a left-hand flag and therefore must also be a left-hand flag. But then the direction assigned to e by $\vec{A}$ and assigned to its preimage in $\vec{B}$ are the same, and so the directions assigned to e by $\vec{A}$ and by $\vec{E}$ are the same. Finally, if the orientations of all of the flags containing the vertex v' (face f') are preserved by α, then the cyclic orientation of the edges around v' (bounding f') must be preserved by α. $\square$

The main result of this section is what follows.

THEOREM 2. *Let ζ be a normal immersion of the unit circle C into $\mathbb{R}^2$. Then ζ admits an extension to an immersion δ of the closed disk $\overline{D}$ into $\mathbb{R}^2$ if and only if the patch $\vec{\Gamma}_\zeta$ admits a covering patch Φ_δ. Furthermore, ζ admits m nonequivalent extensions if and only if $\vec{\Gamma}_\zeta$ admits m nonequivalent covering patches.*

Before we prove this result, we introduce some notation that we will then use throughout the rest of the paper. Let ζ be a normal immersion of the unit circle C into $\mathbb{R}^2$, and let $\vec{\Gamma}_\zeta = (V, \vec{E}, F, \vec{A})$ be the normal patch associated with S the image curve of ζ. By definition, the boundary of any covering patch for $\vec{\Gamma}_\zeta$ must be its straight-through Euler circuit. Let $\vec{\Delta} = (U, \vec{B})$ be the straight-through Euler circuit for $\vec{\Gamma}_\zeta$. For each vertex $v \in V$, there are two vertices in U that map onto v. We label them v_0 and v_1 so that the image of the edge leaving v_1 follows the image of the edge leaving v_0 in the clockwise ordering of the edges incident with v. We illustrate this in Figure 12 (left) and then with the GHZ example. Given the orientation induced on the edges of $\vec{\Gamma}_\zeta$ by $\vec{B}$, the two edges leaving a go to c and f. Since the edge going to f precedes the edge going to c in the clockwise ordering of the four edges incident with a, the initial vertex of the preimage of the a-f edge is label a_1 and the initial vertex of the preimage of the a-c edge is label a_0. The remaining vertices

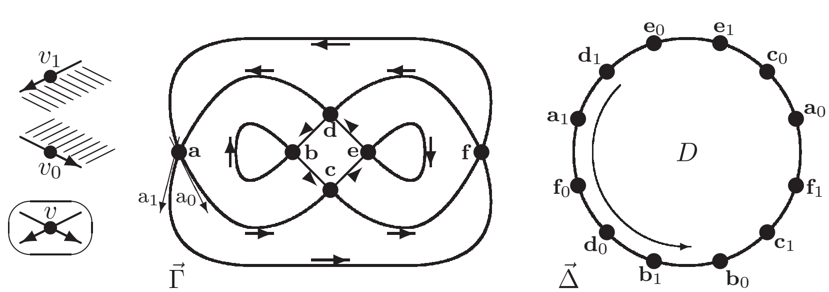

FIG. 12.

in U are labeled in the same way. We will refer to this as the *canonical labeling* of $\vec{\Delta} = (U, \vec{B})$.

It is interesting to note that the map $\vec{\Gamma} = (V, \vec{E}, F')$ ($\vec{\Gamma}_\zeta$ with the outside face reincluded) is uniquely determined as *a map on the sphere* by the directed map $\vec{\Delta}$. One simply starts by picking any edge (x_i, y_j) of $\vec{\Delta}_\zeta$ and drawing its image (x, y) in the plane. Then the next edge (y_j, z_k) is drawn as (y, z) and indicating the direction in which the curve will cross itself at y by an arrow: left to right if $j = 0$; right to left if $j = 1$. One continues in this way until a vertex label is repeated with the second subscript at which time one takes care to draw the curve so that it crosses itself in the appropriate direction. One continues drawing the segments in order. The fact that the crossings can always be made in the correct direction follows from the existence of S. (An arbitrarily labeled circuit may not correspond to an actual curve.) The only problem with this drawing is that any face could turn out to be the outside face. This fact that the crossing sequence uniquely determines the spherical map was first noted by Adkisson and MacLane [1]. We now turn to the proof of the main result of this section.

Proof. Let ζ be a normal immersion of the unit circle C into $\mathbb{R}^2$ with the normal curve S as its image and $\vec{\Gamma}_\zeta = (V, \vec{E}, F, \vec{A})$ as the corresponding normal patch. Assume that $\Phi = (V', E', F', \vec{B})$ is a covering patch for $\vec{\Gamma}_\zeta = (V, \vec{E}, F, \vec{A})$ given by the immersion α. Since the union of the vertices, edges, and faces of the patch Φ is homeomorphic to a disk, we can, without loss of generality, assume that Φ is embedded in the unit disk so that the vertices of Φ on its boundary $(U, \vec{B})$ is the canonical straight-through Euler path for $\vec{\Gamma}_\zeta$. We may assume that the vertices of Φ in U have been assigned the canonical labels. Hence each vertex v of $\vec{\Gamma}_\zeta$ has the same two preimages under ζ^{-1} and α^{-1} on the boundary of Φ (labeled v_0 and v_1). All other preimages of v under α^{-1} lie interior to the disk, and we may assume that they are $v_2, \dots$. We start to construct δ, our extension of ζ to the entire disk by defining $\delta = \zeta$ on the boundary and defining $\delta(v_i) = v$ for each interior vertex of Φ.

By the usual meaning of the term "drawing" the curve of the drawing corresponding to an edge is a homeomorphic image of the unit interval. Hence, the interior of the curve corresponding to a nonboundary edge (u_i, v_j) of Φ is homeomorphic to the open arc on S corresponding to the edge (u, v) in $\vec{\Gamma}_\zeta$. Note that edges of the form (v_i, v_j) map onto loops in $\vec{\Gamma}_\zeta$. We have extended ζ to a local homeomorphism of the drawing of the graph (V', E') onto the curve S. In particular, the boundary of a face f of Φ is mapped by δ onto a closed "subcurve" of S. But since Φ is a covering patch for $\vec{\Gamma}_\zeta$, that subcurve actually bounds the region, R_f, of $\mathbb{R}^2 - S$ corresponding to the face of $\vec{\Gamma}_\zeta$ on which f is projected. Then δ restricted to the boundary of the disk-like region R'_f, corresponding to f in our drawing of Φ, is mapped onto the boundary of the disk-like region R_f. We may then extend δ to a homeomorphism between the interiors of these regions. Carrying out this extension for all faces yields the local homeomorphism δ, the required extension of ζ to the entire disk. If Φ and Φ' are equivalent covering patches for $\vec{\Gamma}_\zeta$, one easily extends the patch isomorphism to a homeomorphism of $\overline{D}$ to itself that fixes C pointwise. Thus the extensions, δ and δ', constructed from Φ and Φ' are also equivalent.

Now let ζ be a normal immersion of the unit circle C into $\mathbb{R}^2$ with the normal curve S as its image and $\vec{\Gamma}_\zeta = (V, E, F, \vec{A})$ as the corresponding normal patch. Assume that ζ has an extension, the immersion δ, mapping the entire disk into $\mathbb{R}^2$. We must use δ to construct a covering patch $\Phi = (V', E', F', \vec{B})$ for $\vec{\Gamma}_\zeta$. We start our construction with the canonical straight-through Euler circuit $\vec{\Delta} = (U, \vec{B})$ (as pictured in Figure 12). Before we construct the remaining vertices and edges of Φ, we construct a special collection of neighborhoods for $\overline{D}$ and its image.

Since δ is a local homeomorphism, we have for each point $x \in \overline{D}$ a neighborhood N_x so that δ restricted to N_x is a homeomorphism into $\mathbb{R}^2$. It will be convenient to alter these neighborhoods so that the neighborhoods for all preimages of the same point $p \in \mathbb{R}^2$ have the same or "corresponding" images under δ. If p does not lie on the curve S, then all preimages $p_0, p_1, \ldots, p_t$ of p lie interior to the disk. Without loss of generality, we may assume that the N_{p_i} are disjoint. Now let $N = \cap_{i=0}^{t} \delta(N_{p_i})$ and then replace N_{p_i} by $\delta^{-1}(N) \cap N_{p_i}$. For points on S and, in particular, for the points of self-intersection, a similar but more complicated-to-describe construction may be employed to produce the "corresponding" neighborhood systems pictured in Figure 13. As indicated in Figure 13, points on the boundary have one or two "half-disk-like" neighborhoods. Note that, for the points of self-intersection, we have used the canonical labeling.

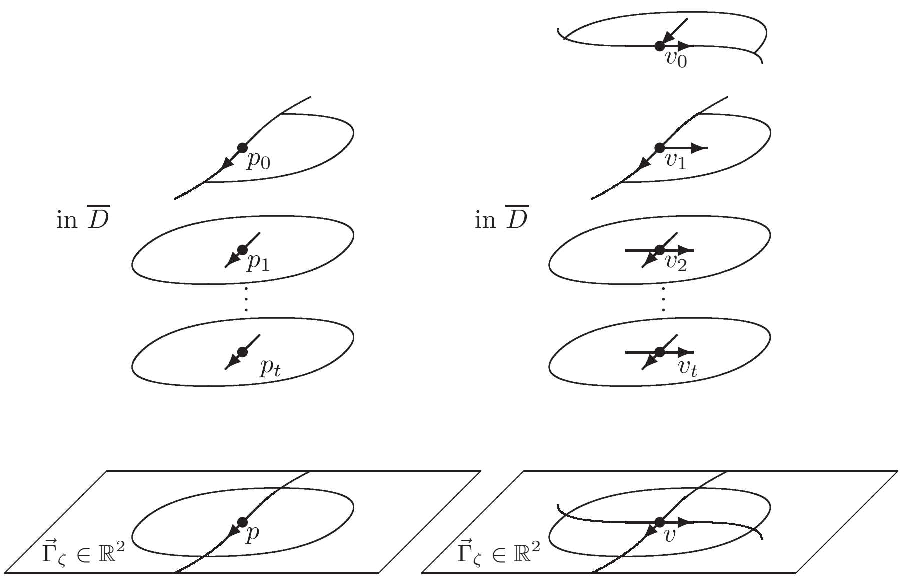

FIG. 13.

Assume that we have replaced all of the N_x in $\overline{D}$ by these corresponding neighborhoods. Since $\overline{D}$ is compact, its image under δ is compact, and a finite collection of the image neighborhoods cover the image. The corresponding neighborhoods of this family then cover $\overline{D}$. We will call this a *uniform collection of neighborhoods*. So far in the construction of $\Phi = (V', E', F', \vec{B})$, we have constructed the boundary and straight-through Euler circuit $(U, \vec{B})$ with immersion maps $\alpha : U \to V$ and $\alpha : \vec{B} \to \vec{E}$. For the vertex set V', we simply take the preimages under δ^{-1} of the points of self-intersection of S (the vertex set of $\vec{\Gamma}_\zeta$) and extend the immersion map already defined on U by defining $\alpha(v_i) = v$ for all v_i.

Now select an edge (u, v) in $\vec{\Gamma}_\zeta$, and let $S_{(u,v)}$ denote the arc of S corresponding to that edge. Using standard constructions, we may select a sequence of points $p_0 = u, p_1, \ldots, p_{m-1}, p_m = v$ on $S_{(u,v)}$ and a corresponding sequence of neighborhoods from our uniform collection, $N_1, \ldots, N_m$, so that the subarc $S_{(p_{i-1},p_i)} \subset N_i$. Next, select u_k, any one of the points in $\overline{D}$ that is mapped onto u. Let M_1 denote the neighborhood in our uniform collection that contains u_k and maps homeomorphically onto N_1. We proceed inductively to define M_{i+1} as the neighborhood in our uniform collection above N_{i+1} that contains the preimage of p_i that lies in M_i. Denote by Π_i the preimage of the arc $S_{(p_{i-1},p_i)}$ in the neighborhood M_i. These short arcs match up in the intersections of successive M_i, producing an arc in $\overline{D}$ joining u_k to some vertex v_j above v - a lifting of the arc $S_{(u,v)}$. Each such lifting identifies an edge (u_k, v_j) joining vertices in V', and E' is the collection of all such edges. One easily checks that boundary edges in $\vec{B}$ are among the edges constructed in this way.

At this point, we have constructed an extension of the directed graph $(U, \vec{B})$ to a graph (V', E') with a drawing in $\overline{D}$, that is, to a patch $\Phi = (V', E', F', \vec{B})$. It is clear that the immersion map $\alpha : E' \to E$ defined by $\alpha((u_k, v_j)) = (u, v)$ and the previously defined immersion map $\alpha : V' \to V$ preserve the graph incidences. All that remains is to extend α to the faces of Φ.

Let f' be a face of the patch Φ; let $x'_1, e'_1, x'_2, \ldots, e'_k, x'_{m+1} = x'_1$ be the boundary circuit of f' ordered clockwise around f'. We first note that as we trace out the boundary of f', the images $\alpha(x'_1), \alpha(e'_1), \alpha(x'_2), \ldots, \alpha(e'_k), \alpha(x'_1)$ trace out the boundary of a face f in $\vec{\Gamma}$. However, before we can define $\alpha(f')$ to be f, we must exclude the possibility that as one traces the boundary of f' one actually traces the boundary of f several times. Suppose that, for some $k < m$, $\alpha(x'_k) = u = \alpha(x'_1)$ and $\alpha(e'_k) = u = \alpha(e'_1)$. Now cover this path in Φ from x'_1 to x'_k by a finite sequence of neighborhoods from our uniform collection. The images of these neighborhoods under δ cover the entire boundary of f. Now add sufficient neighborhoods from the collection to complete a finite cover of f. Now think of the boundary of as a "loop" on u and slowly shrink it back toward u in the face f. As the loop moves from a neighborhood on the boundary to one of the added neighborhoods, select the lifted neighborhood in Φ that intersects the boundary neighborhood in Φ and lift the loop to that neighborhood. Continuing in this way, we maintain a curve in the face f' that maps onto the shrinking loop. Once the loop is small enough to lie entirely in the single neighborhood containing u, its lifted image, a path from x'_1 to x'_k must lie entirely in one of the neighborhoods above u. We conclude that x'_k to x'_m, and we may define $\alpha(f') = f$.

Finally, if δ and δ' are equivalent extensions of ζ, then the equivalence homeomorphism $\beta : \overline{D} \to \overline{D}$ yields a patch isomorphism between the covering patches constructed from δ and δ'. □

5. Constructing covering patches. Let $\vec{\Gamma} = (V, \vec{E}, F, \vec{A})$ be a normal patch. Suppose that $\vec{\Gamma}$ admits a covering patch $\Phi = (V', E', F', \vec{B})$ with the immersion α. It follows from Lemma 1 that the orientation of $\vec{A}$ and the straight-through Euler circuit agree at every edge of the boundary $\vec{A}$. We may assign to each vertex v, edge e, and face f of $\vec{\Gamma}$ its *multiplicity*: $\mu(x) = |\alpha^{-1}(x)|$ for $x = v, e, f$. Consider any vertex of $\vec{\Gamma}$; then α^{-1} consists of v_0 and v_1 on the boundary of Φ and perhaps several other points $\{v_2, \ldots, v_{\mu(v)-1}\}$ "interior" to Φ. By the property (iii) of the definition of immersion and Lemma 1, the neighbors of each of the preimages of v are correctly pictured in Figure 14. Note further that, with the labeling that we have adopted, some labels are missing: d_0 and e'_1 among the lifted edges and g_0, h_0, k_1, and h_1 among the lifted faces.

We observe that

(i) for each $v \in V$, $\mu(v) = \max\{\mu(f) : f \text{ is incident with } v\}$;
(ii) for each $e \in E$, $\mu(e) = \max\{\mu(f) : f \text{ is incident with } e\}$;

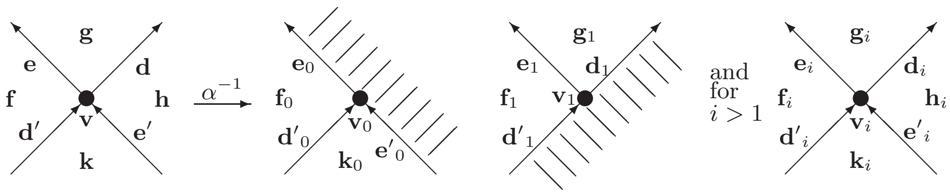

Fig. 14.

(iii) if $e \in E$ and f and g are the faces on the left and right of e as one moves along the Euler circuit, then $\mu(f) = \mu(g) + 1$.

These observations enable us to compute the multiplicities of the covering patch Φ directly for a normal patch $\vec{\Gamma}$. We start with the faces. Assign multiplicity 0 to the outside face, and for any face f, let $f_0, e_1, f_1 e_2, \ldots, e_m, f_m$ be the faces and edges of $\vec{\Gamma}$ corresponding to a dual path from the outside face f_0 to $f = f_m$. Define $\mu(f_i)$ to be $\mu(f_{i-1}) + 1$ if f_i is on the left of e_i and to be $\mu(f_{i-1}) - 1$ if f_i is on the right of e_i. It follows from our observations that whenever $\vec{\Gamma}$ admits a covering Φ, the value of $\mu(f)$ computed in this way is the multiplicity of f under this immersion and, therefore, is independent of the choice of the dual path joining it to the outside face. Actually, there is a straightforward graph theory argument that shows that $\mu(f)$ defined in this way is well defined for any normal patch $\vec{\Gamma}$ whether or not it admits a covering patch.

Let $\vec{\Gamma} = (V, \vec{E}, F, \vec{A})$ denote any normal patch, and let f be any face. Consider two dual paths from o, the outside face to f. Consider one of these dual paths and observe that the number of edges crossing the dual path from left to right minus the number of edges crossing the dual path from right to left is the multiplicity assigned to f by that path. Consider the dual circuit formed by following one of these paths to f and returning to o by the other, as pictured in Figure 15. Then the multiplicity assigned to f by these paths will be the same if and only if traversing this dual circuit in either direction, the number of edges crossing the circuit from left to right equals the number of edges crossing it from right to left.

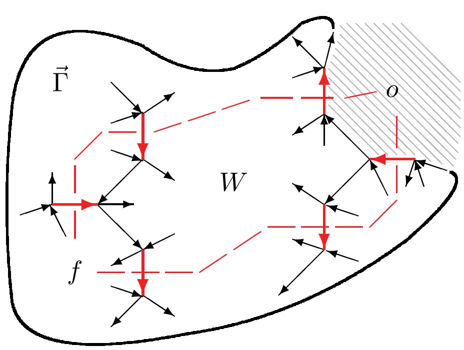

Fig. 15.

Assume that this dual circuit is an elementary circuit, as pictured in Figure 15, and let W denote the collection of vertices it encloses. For each edge with an endpoint $w \in W$ add 1 if it terminates in w and -1 if it initiates at w. This sum is clearly 0 since the contribution at each vertex is two $+1$s and two -1s. On the other hand, each edge crossing the dual circuit from left to right contributes 1 to this sum, each edge crossing the dual circuit from right to left contributes -1 to the sum, and all other edges contribute a $+1$ and a -1 to the sum. The argument is a bit more complicated if the dual circuit is not an elementary circuit, and that case is left to be worked out by the interested reader.

So the *depth-of-cover multiplicity function*, $\mu(f)$, is well defined for the faces of any normal patch $\vec{\Gamma}$. In Figure 16, we have computed μ for the faces of the GHZ example. Once the function μ has been defined on the faces of $\vec{\Gamma}$, we extend it to the vertices and edges by the equations in observations (i) and (ii) above. We note that if $\vec{\Gamma}$ is a normal patch and if $\mu(f) < 0$ for some face, then $\vec{\Gamma}$ does not admit a covering patch.

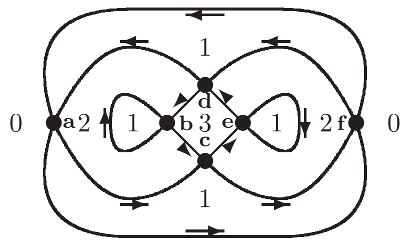

Fig. 16.

Let $\vec{\Gamma} = (V, \vec{E}, F, \vec{A})$ be any normal patch, and let μ denote its multiplicity function. If $\vec{\Gamma}$ admits a covering patch, then that covering patch has $\mu_V = \sum_{v \in V} \mu(v)$ vertices, $\mu_E = \sum_{e \in E} \mu(e)$ edges, and $\mu_F = \sum_{f \in F} \mu(f)$ faces. Furthermore, these numbers must satisfy Euler's equation for planar maps: $\mu_V - \mu_E + (\mu_F + 1) = 2$. Now if a covering patch exists, it has μ_V vertices; $2|V|$ of these have degree 3, and the rest have degree 4. So the sum of the vertex degrees is $4\mu_V - 2|V|$, giving the equation $\mu_E = 2\mu_V - |V|$. Substituting this into Euler's formula and rearranging terms gives $\mu_V - |V| = \mu_F - 1$. As we observed above, at any vertex v of $\vec{\Gamma}$, one of the incident faces has multiplicity $\mu(v)$, two have multiplicity $\mu(v) - 1$, and the remaining face has multiplicity $\mu(v) - 2$ (see Figures 14 and 17). Hence, the sum of the multiplicities of the faces incident with v is $4(\mu(v) - 1)$. Now sum the face multiplicities times the face degrees ($\omega(f)$) to get

$$\sum_{f \in F} \omega(f)\mu(f) = \sum_{v \in V} 4(\mu(v) - 1) = 4(\mu_V - |V|) = 4(\mu_F - 1) = \left(\sum_{f \in F} 4\mu(f) \right) - 4$$

or simply $\sum_{f \in F} (4 - \omega(f))\mu(f) = 4$. We summarize these observations and computations and, for easy reference, include Lemma 1(ii) in Lemma 2 below.

LEMMA 2. *Let $\vec{\Gamma} = (V, \vec{E}, F, \vec{A})$ be a normal patch, and let μ denote its computed multiplicity function. Then if $\vec{\Gamma}$ admits a covering patch, the following conditions must be met:*

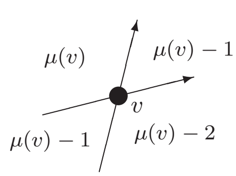

Fig. 17.

(i) *(orientation condition) the orientation of $\vec{A}$ and the orientation of the straight-through Euler circuit agree at every edge of the boundary $\vec{A}$;*
(ii) *(nonnegativity condition) all of the depth-of-cover multiplicities of the faces of $\vec{\Gamma}$ are nonnegative;*
(iii) *(Euler condition) $\sum_{f\in F}(4-\omega(f))\mu(f) = 4$, where $\omega(f)$ denotes the degree of the face f.*

The conditions in Lemma 2 are relatively easy to check for a given patch; one easily checks that each of these conditions is met by the GHZ example. This lemma enables one to quickly eliminate from consideration many normal curves that do not bound a disk. In Figure 18, we illustrate this with two very simple examples. The normal patch based on the normal curve S_1 violates conditions (i) and (ii) of Lemma 2, and while the normal patch based on the normal curve S_2 satisfies conditions (i) and (ii) of Lemma 2, it fails to meet the Euler condition.

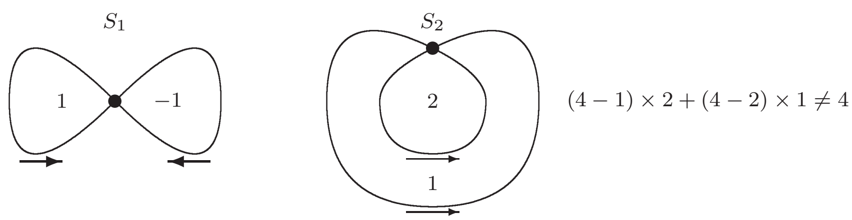

FIG. 18.

Suppose that we have a normal patch $\vec{\Gamma} = (V, \vec{E}, F, \vec{A})$ that satisfies all of the conditions in Lemma 2. We would like to construct all possible covering patches for $\vec{\Gamma}$ or show that there are none. To do this we will closely follow the construction of the covering patch given in the proof of Theorem 2.

Assume that $\vec{\Gamma} = (V, \vec{E}, F, \vec{A})$ is a normal patch that satisfies all of the conditions in Lemma 2. We have already discussed how to construct the boundary circuit $\vec{\Delta} = (U, \vec{B})$. Furthermore, we can construct the vertices of any possible covering patch $\Phi = (V', E', F', \vec{B})$ by simply subscripting the vertices in V: $V' = \bigcup_{v\in V}\{v_0, v_1, \ldots v_{\mu(v)-1}\}$. This gives the first of the immersion maps: $\alpha : V' \to V$ by $\alpha(v_i) = v$ for all $v_i \in V'$. Now consider an edge $e = (u, v)$ of $\vec{\Gamma}$. We know that E' must contain exactly $\mu(e)$ edges projecting onto e. However, as we noted above (Figure 14), it may be the case that $\mu(u) > \mu(e)$ or $\mu(v) > \mu(e)$, and, in that case, not every vertex among $u_0, u_1, \ldots u_{\mu(u)-1}; v_0, v_1, \ldots v_{\mu(v)-1}$ will be the endpoint of an edge in E'. There are four cases to consider, and they are pictured in Figure 19.

We say that an edge $e = (u, v)$ of the normal patch $\vec{\Gamma}$ is of *Type* 00 if e is directed from u to v by the directed Euler circuit and the directed Euler circuit crosses from left to right at both u and v. In Figure 19, we have computed the relative multiplicities of the faces adjacent to the endpoints of e (where h is at least 2). From these face multiplicities, we then compute $\mu(u) = h$, $\mu(v) = h + 1$, and $\mu(e) = h$. So, some vertex above v is not the endpoint of an edge above e. The vertex labeled v_1 lies on the boundary $\vec{B}$ and corresponds to a degree 3 vertex of Φ and the "missing" edge corresponding to e. See Figure 13. Hence, in this case, the edges projecting onto e have one endpoint in $\{u_0, u_1, \ldots, u_{h-1}\}$ and the other endpoint in $\{v_0, v_2, \ldots, v_h\}$. To satisfy condition (iii) of the definition of an immersion, we must insist that each vertex among $\{u_0, u_1, \ldots, u_{h-1}\}$ and $\{v_0, v_2, \ldots, v_h\}$ is the endpoint of just one edge above e. Hence the edges to be mapped onto e correspond to a matching or bijection

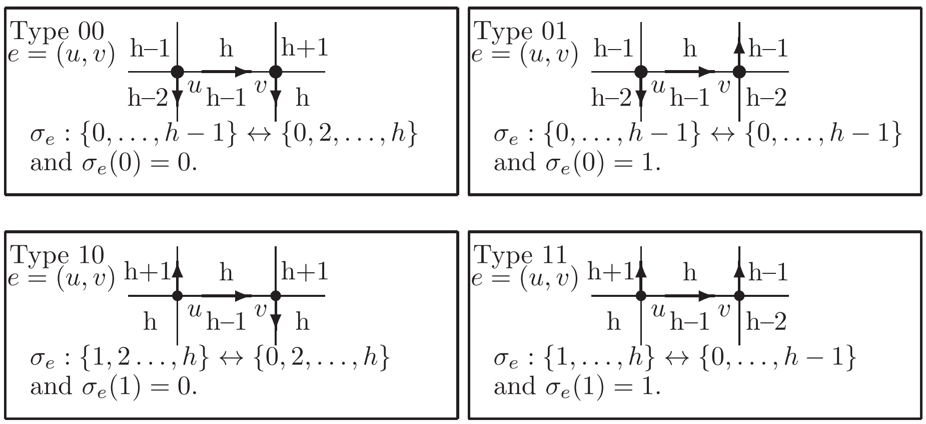

Fig. 19.

between $\{u_0, u_1, \ldots, u_{h-1}\}$ and $\{v_0, v_2, \ldots, v_h\}$. It is convenient to use the notation σ_e to denote the bijection between the sets of subscripts of the relevant vertices above the endpoints of e. Furthermore, in this particular case, the edge in the boundary B mapping onto e goes from u_0 to v_0; so $\sigma_e(0) = 0$. We also note that the edge (u_0, v_0) bounds only a face that projects onto the face of $\vec{\Gamma}$ to the left of e; that is, the faces adjacent to e on the side with the largest multiplicity.

Similarly, we define an edge $e = (u, v)$ to be of *Type* 01 to denote if e is directed from u to v by the directed Euler circuit and the directed Euler circuit crosses from left to right at u and right to left at v. Here, the relevant matching is $\sigma_e : \{0, \ldots, h-1\} \leftrightarrow \{0, \ldots, h-1\}$ with $\sigma_e(0) = 1$. An edge $e = (u, v)$ is of *Type* 10 if e is directed from u to v by the directed Euler circuit and the directed Euler circuit crosses from right to left at u and left to right at v. Here, the relevant matching is $\sigma_e : \{1, \ldots, h\} \leftrightarrow \{0, 2 \ldots, h\}$ with $\sigma_e(1) = 0$. An edge $e = (u, v)$ is of *Type* 11 if e is directed from u to v by the directed Euler circuit and the directed Euler circuit crosses from right to left at both u and v. Here, the relevant matching is $\sigma_e : \{1, \ldots, h\} \leftrightarrow \{0, \ldots, h-1\}$ with $\sigma_e(1) = 1$.

Suppose that we simply complete the definitions of the matching σ_e in some way for every edge. We would have constructed a graph (V', E') that is an extension of $(U, \vec{B})$ and has the required immersion functions $\alpha : V' \to V$ and $\alpha : E' \to E$. In addition for each vertex $v_i \in V'$, there is a natural cyclic ordering of the edges around v_i, namely, the counterclockwise ordering of their images around v in $\vec{\Gamma}$. If we add the condition that the resulting graph (V', E') is connected, we may apply the Edmonds' embedding technique [4] as follows.

Let (V, E) be any connected graph and assign to each vertex a cyclic ordering of the edges with it as endpoint. Now construct the boundaries of the faces of an embedding as follows: start with any initial vertex x_1 and incident edge e_1, move along the edge to the other endpoint x_2 and let e_2 be the next edge in the cyclic ordering around x_2 after e_1, and so on. Edmonds proved that this process will end with $x_h = x_1$ and $e_h = e_1$ resulting in a directed circuit. He then proved that the set of directed circuits produced in this way are the clockwise-oriented boundaries of the faces of an embedding of (V, E) in some orientable surface.

With the Edmonds construction in mind, we require that the completions of the definitions of the matchings above the edges satisfy the following conditions:

(i) the resulting graph is connected;
(ii) the composition of the matchings as one traverses the boundary of a face of $\vec{\Gamma}$ is the identity matching.

We say that such a collection of bijections is *acceptable*. Condition (ii) implies that the set of faces constructed by the Edmonds technique consists of one face with boundary B and a collection F' of faces that map down to the faces of $\vec{\Gamma}$, giving the final immersion function $\alpha : F' \to F$. Finally, we note that from the Edmonds result we can only conclude that the surface on which $\Phi = (V', E', F', \vec{B})$ is embedded is orientable. However, we started with a normal patch that satisfied the Euler condition (Lemma 2(ii)). So the the vertex, edge, and face numbers of Φ satisfy the Euler formula for a planar embedding, and we may conclude that Φ is actually planar. We have proved part (i) of Theorem 3.

Theorem 3. *Let $\vec{\Gamma}$ be a normal patch.*
(i) *$\vec{\Gamma}$ admits a covering patch if and only if it satisfies the conditions in Lemma 2 and admits an acceptable collection of bijections.*
(ii) *Two covering patchs $\Phi = (V', E', F', \vec{B})$ and $\Phi^* = (V', E^*, F', \vec{B})$, given by the acceptable collections of bijections σ_e and σ_e^*, are equivalent if and only if there exists a family of permutations $\pi_v : \{v_2, \ldots, v_h\}$, for each $v \in V$, so that $\sigma_e^* = \pi_v \sigma_e = \pi_u$ for each edge $e = (u, v) \in \vec{E}$.*

Proof of Theorem 3 (ii). Given a covering patch $\Phi = (V', E', F', \vec{B})$ for $\vec{\Gamma} = (V, \vec{E}, F, \vec{A})$, it is clear that simply permuting the indices of $\{v_2, \ldots, v_h\}$ for each $v \in V$ will result in an equivalent covering patch. Now suppose that the covering patchs $\Phi = (V', E', F', \vec{B})$ and $\Phi^* = (V', E^*, F', \vec{B})$, given by the acceptable collections of bijections σ_e and σ_e^*, are equivalent. Let α and α^* denote the immersions of Φ and Φ^* onto $\vec{\Gamma}$, and let β denote the homeomorphism from Φ onto Φ^*. Since $\alpha = \beta\alpha^*$, β must map the preimages of a vertex v onto the preimages of v, and since β must fix the boundary, β simply permutes the sets of preimages in $\{v_2, \ldots, v_h\}$ for each $v \in V$. $\square$

We illustrate this result with the GHZ example.

We note first that a and f have multiplicity 2, while b, c, d, and e have multiplicity 3. Hence the vertices of any lifting are

$$\{a_0, a_1, b_0, b_1, b_2, c_0, c_1, c_2, d_0, d_1, d_2, e_0, e_1, e_2, f_0, f_1\}.$$

Next we note that the matchings for edges with multiplicities 1 ((a,f) and (f,a)) or 2 ((d,a), (a,c), (b,b), (c,f), (f,d), and (e,e)) are all forced, as illustrated in Figure 20. In that figure, we diagram the bijections around each of the seven inside faces of $\vec{\Gamma}$. For each face, we list the vertices in clockwise order around the face, draw in the directed edges, and list their types above them. Then, below the heavy horizontal line, we have filled in the assignments that are required by the edge type as bold face arrows, and any additional assignments that are then forced are indicated by lighter arrows. The vertices that are relevant to the matchings around that face are circled.

It remains to discover just how many (if any) different ways that these forced assignments may be completed to an acceptable collection of bijections. But first, we verify that the assignments we claim to be forced are indeed forced. Consider the top face of depth 1. Referring to Figure 19, we see that $\sigma_{(f,a)}$ maps {1} to {0}, and, therefore, (f_1, a_0) is the only lifting of the edge (f, a). Next, $\sigma_{(f,d)}$ maps {0,1} to {0,2}, and $\sigma_{(f,d)}(0) = 0$ is mandated; hence $\sigma_{(f,d)}(1) = 2$, giving (f_0, d_0) and (f_1, d_2) as the two liftings of (f, d). Finally, $\sigma_{(d,a)}$ maps {1,2} to {0,1} with $\sigma_{(d,a)}(1) = 1$ required; hence $\sigma_{(d,a)}(2) = 0$, giving (d_1, a_1) and (d_2, a_0) as the two liftings of (d, a).

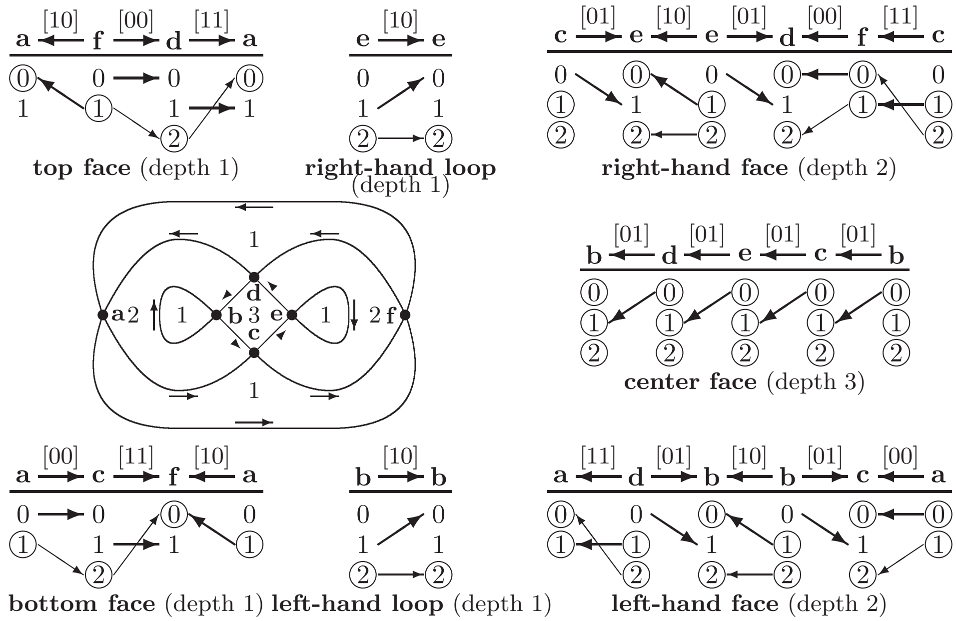

FIG. 20.

These bijections restricted the circled indices (the indices relevant to the face under consideration) compose the identity around this face. Considering the right-hand loop, we have that $\sigma_{(e,e)}$ maps {1,2} to {0,2} and that $\sigma_{(e,e)}(1) = 0$ is mandated; hence $\sigma_{(e,e}(2) = 2$, giving (e_1, e_0) and (e_2, e_2) as the two liftings of (e, e) and (e_2, e_2) as the boundary of the lifting of this face. The arguments for the bottom face of depth 1 and the left-hand loop are the same and result in the edges given in their diagrams. These mandated and induced edges have been filled in on the diagrams for the faces with multiplicities of 2 and 3, and we proceed to the problem of completing the bijections.

Now consider the right-hand face of depth 2 and the edge (c, e). We have two options (c_1, e_0) and (c_2, e_2) or (c_1, e_2) and (c_2, e_0). But once this choice is made, the matching above the edge (e, d) is forced by the condition that the composition of the bijections around the face be the identity. So our two options for choice 1 become the following:

Option 1A. (c_1, e_0), (c_2, e_2), (e_1, d_2), and (e_2, d_0);

Option 1B. (c_1, e_2), (c_2, e_0), (e_1, d_0), and (e_2, d_2).

Similarly, the left-hand face of depth 2 yields exactly two options:

Option 2A. (d_1, b_0), (d_2, b_2), (b_1, c_2), and (b_2, c_0);

Option 2B. (d_1, b_2), (d_2, b_0), (b_1, c_0), and (b_2, c_2).

We must now check which of these pairs of options work around the face of depth 3. If we select Option 1A and track the trajectory of b_0, we conclude that we must then select Option 2A. One easily checks that this pair of options work and yield the covering patch in Figure 10 (left). If we select Option 1B and again track the trajectory of b_0, we conclude that we must select Option 2B. Again, one easily checks that this pair of options work and yield the covering patch in Figure 10 (right). Finally, since there are not enough indices above 1 to permute, these covering patches are not equivalent.

It follows from our analysis as illustrated in Figure 19 that, in general, the matching above every edge of multiplicity 1 or 2 is uniquely determined. Thus, if a normal curve S with maximum multiplicity 1 or 2 bounds a distorted disk, that disk is unique up to equivalence. Hence, we have the continuous analogue to main result of [5].

COROLLARY 3.1. *If the normal curve S bounds two distinct distorted disks, then it must have some face with a multiplicity of at least* 3.

When we apply the theorem to normal curves with maximum multiplicity of 2, we are able to simplify the necessary and sufficient conditions for the existence of a covering patch. Consider a piece of the curve containing an edge (u, v) bounding a face of multiplicity 2, as pictured in Figure 21. We argue that the segment crossing our segment at u must cross from left to right; otherwise the face labeled f would have depth 3. Also the crossing at v must be from right to left; otherwise the face labeled g would have depth 3. We conclude that (x, y) is of type [01]. Hence we have no choice in defining the bijections: for every edge e of multiplicity 2, $\sigma_e(0) = 1$ and $\sigma_e(1) = 0$. Now consider the composition of matchings around any face of multiplicity 2. Each matching reverses the indices; hence composition will be the identity if and only if the number of matchings is even, i.e., if and only if every face of multiplicity 2 has even degree.

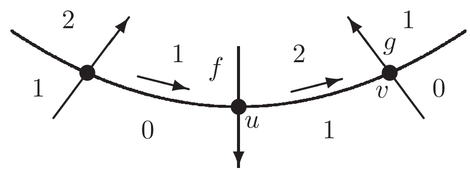

FIG. 21.

With the aid of Figure 21, we may simplify the Euler condition (condition (iii) of Lemma 2). Let F_i denote the faces of $\vec{\Gamma}$ of multiplicity i and include the outside face in F_0. Each vertex of $\vec{\Gamma}$ is incident with one face in F_0, one in F_2, and two in F_1. This gives $\sum_{f \in F_0} \delta(f) = \sum_{f \in F_2} \delta(f) = |V|$, while $\sum_{f \in F_1} \delta(f) = 2|V|$. Plugging these values into condition (iii) of Lemma 2 gives the equivalent condition

$$4|F_1| - 2|V| + 8|F_2| - 2|V| = 4 \quad \text{or} \quad 2|F_2| + |F_1| = |V| + 1.$$

We may further simplify this condition by applying Euler's formula directly to $\vec{\Gamma}$. Since every vertex of $\vec{\Gamma}$ has degree 4, $|\vec{E}| = 2|V|$, and by Euler's formula

$$|V| - 2|V| + (|F_0| + |F_1| + |F_2|) = 2.$$

Combining these two equations to eliminate $|F_1|$, we see that $\vec{\Gamma}$ will satisfy condition (iii) of Lemma 2 if and only if $|F_0| = |F_2| + 1$ or the number of faces of $\vec{\Gamma}$ of multiplicity 2 equals the number of internal faces of $\vec{\Gamma}$ of multiplicity 0. We have proved the next corollary.

COROLLARY 3.2. *Let $\vec{\Gamma}$ be a properly oriented normal patch with nonnegative multiplicities (i.e., satisfying conditions* (i) *and* (ii) *of Lemma* 2*) and having maximum multiplicity of* 2*. Then $\vec{\Gamma}$ admits a covering patch if and only if*

(i) *the number of faces with multiplicity* 2 *equals the number of internal faces with multiplicity* 0*.*

(ii) *all faces of $\vec{\Gamma}_S$ with multiplicity* 2 *have boundaries of even length.*

Furthermore, if a covering patch exists, it is unique.

Applying Corollary 3.2 to the curves in Figure 22, we see immediately that both curves give properly oriented normal patches with nonnegative multiplicities and that both satisfy condition (i) of the corollary. But only the curve on the right satisfies condition (ii), and hence only the curve on the right bounds a distorted disk.

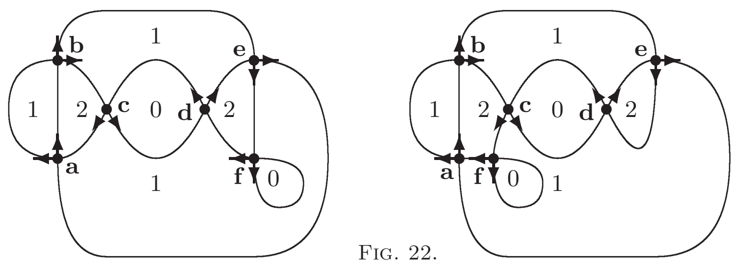

FIG. 22.

6. Covering the sphere. Consider $\zeta : C \to \mathbb{S}^2$, and let its image S be a normal curve drawn on the sphere. Here there is no outside face, and so we can associate with the directed curve the directed multigraph $\vec{\Gamma}_S = (V, \vec{E}, F)$, where V is the set of crossing points of S, $\vec{E}$, the directed arcs between crossing points, and F the components of $\mathbb{S}^2 - S$. Suppose that ζ may be extended to an immersion $\delta : \overline{D} \to \mathbb{S}^2$.

As in the planar case, this leads to a covering patch $\Phi = (V', E', F', \vec{B})$. The development of the structure of covering patches for spherical multigraphs parallels the above development to the structure of covering patches for patches. In the following development, proofs that are parallel to the proofs of the corresponding planar result are omitted; we concentrate on the few differences between and immersions into the plane and immersions onto the sphere.

We start by making the following definitions. By a *normal multigraph* we shall mean a directed, 2-connected, plane multigraph $\vec{\Gamma} = (V, \vec{E}, F)$ such that

(i) each vertex of $\vec{\Gamma}$ has degree 4,

(ii) $\vec{\Gamma}$ admits a straight-through Euler circuit $\vec{\Delta} = (U, \vec{B})$.

An *immersion* α of a patch $\Phi = (V', E', F', \vec{B})$ onto a normal multigraph $\vec{\Gamma} = (V, \vec{E}, F)$ is a collection of onto functions (all using the same symbol): $\alpha : V' \to V$, $\alpha : E' \to E$, $\alpha : F' \to F$ such that

(i) all incidences are preserved;

(ii) α restricted to $(U, \vec{B})$, where U is the set of vertices on the boundary, is the straight-through Euler circuit for $\vec{\Gamma}$;

(iii) for each $v' \in V'$, α restricted to v' and the set of edges and faces incident with v' is one to one and onto;

(iv) for each $f' \in F'$, α restricted to f' and the set of edges incident with f' is one to one and onto.

We say that Φ is a *covering patch* of the normal multigraph $\vec{\Gamma}$ if there is an immersion α of Φ onto $\vec{\Gamma}$.

With these slightly altered definitions, the proofs of Lemma 1 and Theorem 2 are easily adapted to prove their spherical analogues.

LEMMA 3. *Patch immersions onto normal multigraphs preserve flag orientation and the cyclic orientations of edges around each vertex and face.*

THEOREM 4. *Let ζ be a normal immersion of the unit circle C into $\mathbb{S}^2$. Then ζ admits an extension to an immersion δ of the closed disk $\overline{D}$ into $\mathbb{S}^2$ if and only if the normal multigraph $\vec{\Gamma}_\zeta$ admits a covering patch Φ_δ. Furthermore, ζ admits m nonequivalent extensions if and only if $\vec{\Gamma}_\zeta$ admits m nonequivalent covering patches.*

So we now turn to the problem of constructing all covering patches for a given normal multigraph on the sphere or showing that there are none. But before going any further we note that, because of the outside face in the planar case, only one orientation of the curve S could possible lead to it bounding a distorted disk. That is no longer true on the sphere. So we may wish to consider the possibility of reversing the direction of S, that is, all directions in the normal multigraph $\vec{\Gamma} = (V, \vec{E}, F)$. We will denote this reversed direction normal multigraph by $\vec{\Gamma}_R = (V, \vec{E}_R, F)$.

The first difficulty that we encounter on the sphere is in the computation of the depth-of-cover multiplicities. In fact, the depth-of-cover multiplicities are not well defined for the normal multigraph $\vec{\Gamma}$. The problem is that there is no obvious choice for the outer face. Since there is no outer face, we simply pick some face and arbitrarily assign it the multiplicity 0. Then moving from face to face as before, we compute the depth-of-cover multiplicities for the remaining faces. The combinatorial argument given earlier shows that these multiplicities are well defined. Also it is clear from that argument that if we had started with p instead of 0, the resulting multiplicities can be obtained by simply adding p to each of the multiplicities computed by starting at 0.

Clearly the multiplicities assigned to $\vec{\Gamma}$ so that the least multiplicity is 0 are uniquely determined; we call these the canonical multiplicities for $\vec{\Gamma}$. The *multiplicity gap* or simply *gap* for $\vec{\Gamma}$ is $g(\vec{\Gamma})$, the maximum multiplicity among the canonical multiplicities for $\vec{\Gamma}$. Now consider the canonical multiplicities for both $\vec{\Gamma}$ and $\vec{\Gamma}_R$. Consider the same sequence of faces in $\vec{\Gamma}$ and $\vec{\Gamma}_R$. As one moves through this sequence, multiplicities increase in $\vec{\Gamma}^R$ when they decrease in $\vec{\Gamma}$ and decrease in $\vec{\Gamma}_R$ when they increase in $\vec{\Gamma}$ (see Figure 23 below). These observations lead to the following lemma.

LEMMA 4. *Let $\vec{\Gamma} = (V, \vec{E}, F)$ be a normal multigraph. Let $\mu : F \to \mathbb{N}$, where $\mathbb{N}$ denotes the nonnegative integers, be its canonical assignment of depth-of-cover multiplicities, and let $\mu_R : F \to \mathbb{N}$ be the canonical assignment of multiplicities for $\vec{\Gamma}_R$. Then*

(i) *if μ' is any set of multiplicities for $\vec{\Gamma}$, then there is a constant c so that $\mu'(f) = \mu(f) + c$ for all $f \in F$.*
(ii) *$\vec{\Gamma}$ and $\vec{\Gamma}_R$ have the same gap g,*
(iii) *$\mu_R(f) = g - \mu(f)$ for all $f \in F$.*

Next in this investigation for the sphere, using exactly the same proof, we get the analogue to Lemma 2.

LEMMA 5. *Let $\vec{\Gamma} = (V, \vec{E}, F)$ be a normal multigraph, and let μ denote one of its multiplicity functions. Then if $\vec{\Gamma}$ admits a covering patch with these multiplicities, the following conditions must be met:*

(i) *(nonnegativity condition) all of the multiplicities of the faces of $\vec{\Gamma}$ are nonnegative;*
(ii) *(Euler condition) $\sum_{f \in F}(4 - \delta(f))\mu(f) = 4$.*

The conditions in Lemma 5 are relatively easy to check for a given patch; one easily checks that each of the assignments of multiplicities in Figure 23 satisfies the first of these conditions but that the second condition is met only by the assignment C.

Since we have some flexibility in choosing the multiplicities for a given normal multigraph and its reverse, it is natural to ask how the sum in Lemma 5(ii) changes as we change multiplicities. For the possible sets of multiplicities μ, we define the function $k(\mu) = \sum_{f \in F}(4 - \delta(f))\mu(f)$.

LEMMA 6. *Let $\vec{\Gamma} = (V, \vec{E}, F)$ be a normal multigraph, let μ and μ_R denote the canonical multiplicity functions for $\vec{\Gamma}$ and $\vec{\Gamma}_R$, respectively, let g denote their common gap, and let $\mu' = \mu + c$ for some $c > 0$. We have*

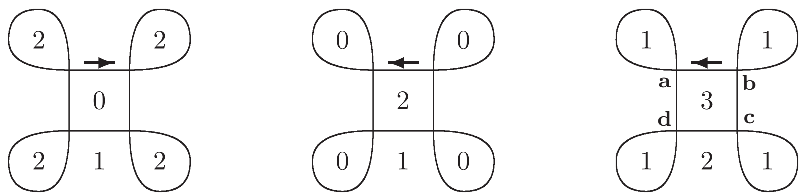

FIG. 23.

(i) $k(\mu') = k(\mu) + 8c$.
(ii) $k(\mu_R) = 8g - k(\mu)$.

Furthermore, among all of the sets of multiplicities for $\vec{\Gamma}$ and $\vec{\Gamma}_R$, at most one set can satisfy the condition in Lemma 5(ii).

Proof. The key to this result is the fact that $\sum_{f\in F}(4-\delta(f)) = 8$:

$$\sum_{f\in F}(4-\delta(f)) = 4|F| - \sum_{f\in F}(\delta(f)) = 4|F| - 2|\vec{E}|.$$

Now from Euler's formula applied to $\vec{\Gamma}$, $4|V| - 4|\vec{E}| + 4|F| = 8$, and since each vertex has degree 4, $4|V| = 2|\vec{E}|$. Thus $4|F| - 2|\vec{E}| = 8$. We have

(i) $k(\mu') = \sum_{f\in F}(4-\delta(f))(\mu(f)+c) =$
$= k(\mu) + \sum_{f\in F}(4-\delta(f))c = k(\mu) + 8c.$
(ii) $k(\mu) + k(\mu_R) = \sum_{f\in F}(4-\delta(f))\mu(f) + \sum_{f\in F}(4-\delta(f))\mu_R(f) =$
$= \sum_{f\in F}(4-\delta(f))g = 8g.$

It is clear from (i) that no two distinct multiplicities for $\vec{\Gamma}$ (or $\vec{\Gamma}_R$) could both satisfy the condition in Lemma 5(ii). Now suppose that $\mu' = \mu + c$ and $\mu'_R = \mu_R + c_R$ both satisfy the condition in Lemma 5(ii). Then $k(\mu') + k(\mu'_R) = 8$. On the other hand, $8 = k(\mu') + k(\mu'_R) = k(\mu) + k(\mu_R) + 8c + 8c_R = 8(g + c + c_R)$. The only solution is $c = c_R = 0$ and $g = 1$, which is impossible. (One could interpret this as the parameters of a single non-self-intersecting curve where either orientation bounds a hemisphere.) ∎

This result is also illustrated by the graphs in Figure 23.

We construct a covering patch for normal multigraph on the sphere in exactly the same way that we construct a covering patch for normal patch in the plane. The theorem for the sphere analogues to Theorem 3 has almost the same statement and exactly the same proof.

THEOREM 5. *Let $\vec{\Gamma}$ be a normal patch.*

(i) *$\vec{\Gamma}$ admits a covering patch if and only if it admits a set of multiplicities that satisfy the conditions in Lemma* 5 *and it admits an acceptable collection of bijections.*
(ii) *Two covering patchs $\Phi = (V', E', F', \vec{B})$ and $\Phi^* = (V', E^*, F', \vec{B})$, given by the acceptable collections of bijections σ_e and σ_e^*, are equivalent if and only if there exists a family of permutations $\pi_v : \{v_2, \dots, v_h\}$, for each $v \in V$, so that $\sigma_e^* = \pi_v \sigma_e = \pi_u$ for each edge $e = (u,v) \in \vec{E}$.*

The curve in Figure 23 is an example of a curve S on the sphere that bounds a distorted disk. To satisfy the conditions of Lemma 5 we must the choose the multiplicities assigned in C. There are three families of bijections that compose the identity around each face. But one results in a disconnected graph, leaving two acceptable collections of bijections. They are isomorphic, but there is no isomorphism that fixes the boundary. They yield the two distinct extensions pictured in Figure 24. We note that these distorted disks cover the entire sphere. It follows that this curve embedded in the plane does not bound a distorted disk.

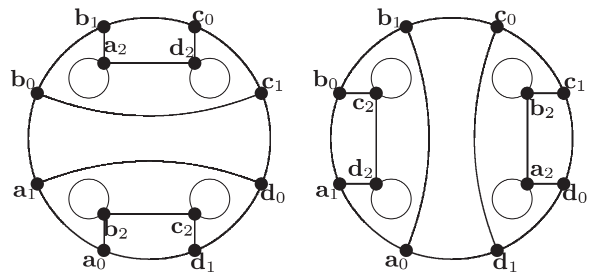

FIG. 24.

7. Comments and questions. We note that any normal curve S in the plane can be identified with a normal curve on the sphere. If S bounds a distorted disk in the plane, it also does so on the sphere, but that disk will not cover the entire sphere. Conversely, any normal curve S on the sphere that bounds a distorted disk that does not cover the entire sphere can be identified with a normal curve S in the plane that bounds a distorted disk. Hence we may think of the immersion problem in the plane as a special case of the immersion problem in the sphere.

Recall that Adkisson and MacLane [1] showed that the embedding of the curve in the sphere is uniquely determined up to equivalence by the sequence of crossings, that is, by the code of indexed vertices. The code for the GHZ curve is

$$a_0c_0e_1e_0d_1a_1f_0d_0b_1b_0c_1f_1,$$

and the code for the curve in Figure 23 is

$$a_0d_1d_0c_1c_0b_1b_0a_1,$$

Is there a reasonable algorithm whereby one could compute directly from the code the number of extensions to the sphere and whether or not they are really planar extensions?

Clearly, a catalog of the values of such a function could be computed, perhaps by computer, for all sufficiently short codes. Would such a catalog be of any interest? Finally, is there any easy-to-see connection between these codes and the Blank words discussed above?

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