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A note on natural correspondences that satisfy exclusion

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Let X be a finite set with m elements. A function $F : \mathcal{P}_2(X) \to \mathcal{P}(X)$ (from the collection of all 2-element subsets of X into the collection of all subsets of X) is called a *natural correspondence* if $S \subseteq F(S)$, for all $S \in \mathcal{P}_2(X)$. We say that F satisfies *exclusion* if there exists some 3-element subset $\{x, y, z\}$ such that $x \notin F(\{y, z\})$, $y \notin F(\{x, z\})$ and $z \notin F(\{x, y\})$.

Observation 1 *F does not satisfy exclusion if and only if, for every $T \in \mathcal{P}_3(X)$, there exists some $S \in \mathcal{P}_2(X)$ such that $S \subseteq T \subseteq F(S)$.*

Let $F : \mathcal{P}_2(X) \to \mathcal{P}(X)$ be a natural correspondence; we define the average image size a_F by:

$$a_F = \frac{1}{\binom{m}{2}} \sum_{S \in \mathcal{P}_2(X)} |F(S)|$$

Theorem 1 *Let $F : \mathcal{P}_2(X) \to \mathcal{P}(X)$ be a natural correspondence that does not satisfy exclusion. Then $a_F \geq \frac{m+4}{3}$.*

Proof Since F does not satisfy exclusion, we may assign to each $T \in \mathcal{P}_3(X)$ a subset $S \in \mathcal{P}_2(X)$ such that $S \subseteq T \subseteq F(S)$. Now consider a subset $S \in \mathcal{P}_2(X)$ and let $T_1, \ldots, T_k$ be the 3-element subsets to which S has been assigned. Then $|F(S)| \geq k + 2$. Summing this inequality over all 2-element subsets gives:

$$\sum_{S \in \mathcal{P}_2(X)} |F(S)| \geq \binom{m}{3} + 2\binom{m}{2}.$$

Dividing through by $\binom{m}{2}$ and simplifying gives the result. □

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Corollary 1 *If $F : \mathcal{P}_2(X) \to \mathcal{P}(X)$ is a natural correspondence with $a_F < \frac{m+4}{3}$, then F satisfies exclusion.*

This result is *best possible* in that, for any m, one may construct a natural correspondence $F : \mathcal{P}_2(X) \to \mathcal{P}(X)$ that does not satisfy exclusion for which $a_F = \frac{m+4}{3}$. First, to each $T \in \mathcal{P}_3(X)$, assign a subset $S \in \mathcal{P}_2(X)$ such that $S \subseteq T$. Now, for each subset $S \in \mathcal{P}_2(X)$, let $T_1, \ldots, T_k$ be the collection of 3-element subsets to which S has been assigned and define $F(S) = T_1 \cup \cdots \cup T_k \cup S$. We close with a simple example.

Example 1 Let $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$. We will denote our subsets by their characteristic functions: 11010000 denotes $\{1, 2, 4\}$; in fact since our example will be symmetric under cyclic permutations, 11010000 will denote the sets $\{1, 2, 4\}, \{2, 3, 5\}, \ldots, \{8, 1, 3\}$.

Up to a cyclic permutation, there are only 4 types of 2-element subsets: α, 11000000; β, 10100000; γ, 10010000; δ, 10001000 and 7 types of 3-element subsets: a, 11100000; b, 11010000; c, 11001000; d, 11000100; e, 11000010; f, 10101000; g, 10100100. To each 3-subset of type a or b assign the 2-subset of type β that it contains; to each 3-subset of type c or g assign the 2-subset of type γ that it contains; to each 3-subset of type d or e assign the 2-subset of type α that it contains; finally to each 3-subset of type f assign the 2-subset of type δ that it contains. Then

$$F(11000000) = 11000110, \quad F(10100000) = 11100001,$$
$$F(10010000) = 10010011, \quad F(10001000) = 10101010,$$

and $a_F = 4 = \frac{8+4}{3}$.

Reference

CK Campbell DE, Kelly JS (2004) Social welfare functions that satisfy Pareto, anonymity and neutrality, but not IIA