Abstract Rigidity in m-Space.

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Abstract

Abstract rigidity matroids are generalizations of the infinitesimal rigidity matroids of frameworks in Euclidean space. In this paper we give a local characterization for abstract rigidity in any dimension. The conditions in this characterization are in many instances easier to verify than those in the definition of these matroids.

A *framework* in m-space is a triple $(V, E, \mathbf{p})$, where (V, E) is a graph and $\mathbf{p}$ is an embedding of V into real m-space. Let $V = \{1, 2, \ldots, n\}$. Regarding $\mathbf{p}$ as a point in real mn-space, the distance constraints corresponding to E give a system of $|E|$ quadratic equations in the coordinates of $\mathbb{R}^{mn}$. The solution set of these equations is an algebraic set $\mathcal{A}$ in mn-space called the *configuration space of* $\mathbf{p}$. Clearly $\mathbf{p} \in \mathcal{A}$ and we may describe a physical movement of the framework in space, that is, a movement of the vertices which preserves the lengths of the edges, by a path in $\mathcal{A}$ starting at $\mathbf{p}$. A framework is *rigid* if the points on $\mathcal{A}$ in a neighborhood of $\mathbf{p}$ all correspond to a framework congruent to $(V, E, \mathbf{p})$, i.e. the only motions which preserve all of the lengths of the edges are the direct isometries of $\mathbb{R}^m$. One approach to detecting rigidity is to replace the system of quadratic equations with their derivatives,

$$(\mathbf{u}(i) - \mathbf{u}(j)) * (\mathbf{p}(i) - \mathbf{p}(j)) = 0, \text{ for all } (i, j) \in E \ ,$$

where $\mathbf{u}(i)$ denotes the initial velocity, at $\mathbf{p}(i)$, of a motion of the framework. This system of linear equations represents the condition that, initially, the motion neither stretches nor contracts the edges. We say that $\mathbf{p}$ is *infinitesimally rigid* if the all solutions to this system arise from the infinitesimal isometries of $\mathbb{R}^m$, i.e. if the only solutions to the system are the restrictions to the set $\{\mathbf{p}(i)\}$ of vector valued functions $\mathbf{u}$ on $\mathbb{R}^m$ which satisfy

$$(\mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q})) * (\mathbf{p} - \mathbf{q}) = 0, \text{ for all } \mathbf{p}, \mathbf{q} \in \mathbb{R}^m.$$

Note that these infinitesimal isometries always yield solutions to the the system of equations of the framework; in fact, they form a $\binom{m+1}{2}$ dimensional subspace of the solution space of the system of the framework. Thus, $\mathbf{p}$ is infinitesimally rigid if and only if the dimension of the solution space of the corresponding system is $\binom{m+1}{2}$. Taking a framework on the complete graph (V, K), the resulting linear system may be used to define a matroid $\mathcal{F}(\mathbf{p})$ on K by saying that $E \subseteq K$ is independent if the system of linear equations corresponding to E is independent. E is then infinitesimally rigid if the closure of E in $\mathcal{F}(\mathbf{p})$ is $K(V(E))$, where $V(E)$ denotes the set of vertices of the edges in E and $K(U)$, for any $U \in V$, denotes the set of all edges with both endpoints in U.

More generally, given a finite set V let $K = K(V)$ denote the edge set of the complete graph on V. Then a matroid $\mathcal{A}_m$ on K with closure operator $\langle \cdot \rangle$ is called an *m-dimensional abstract rigidity matroid for* V if, besides the usual axioms for a matroid,

C1 $T \subseteq \langle T \rangle$;

C2 If $R \subseteq T$, then $\langle R \rangle \subseteq \langle T \rangle$;

C3 $\langle \langle T \rangle \rangle = \langle T \rangle$.

C4 If $s, t \in (E - \langle T \rangle)$, then $s \in \langle T \cup \{t\} \rangle$ if and only if $t \in \langle T \cup \{s\} \rangle$.

it satisfies the additional conditions C5 and C6 below:

C5 If $E, F \subseteq K$ and $|V(E) \cap V(F)| < m$, then $\langle E \cup F \rangle \subseteq (K(V(E)) \cup K(V(F)))$.

C6 If $\langle E \rangle = K(V(E))$, $\langle F \rangle = K(V(F))$ and $|V(E) \cap V(F)| \geq m$, then $\langle E \cup F \rangle = K(V(E \cup F))$.

Note that Axiom C5 implies $\langle E \rangle \subseteq K(V(E))$, and, motivated by the infinitesimal case, we say $E \subseteq K$ is *rigid* if $\langle E \rangle = K(V(E))$. With this definition, C6 reads: if two rigid sets have m or more vertices in common, then their union is rigid.

If $\mathbf{p}$ embeds the vertices of a framework in general position, then $\mathcal{F}(p)$ is an abstract rigidity matroid. There exist, however, abstract rigidity matroids which are not infinitesimal, [4], and abstract rigidity may also arise in other contexts, see [3] or [5]. The purpose of this note is to characterize the m-dimensional abstract rigidity matroids.

Let $\mathcal{A}_m$ be an m-dimensional abstract rigidity matroid. If E is any edge set such that $|V(E)| \leq m + 1$ then, by C5, E is independent. In particular, K_{m+1} is independent. If $v \notin V(E)$ and F a set of m or fewer edges joining v to vertices in $V(E)$, then $E \cup F$ is called a 0-*extension of* E. Again by C5, 0-extensions of independent sets are independent. In particular, $K_{m+2} - e$ is a 0-extension of K_{m+1}, and hence independent. Moreover, $K_{m+2} - e$ is the union of two K_{m+1}'s along a K_m, and so by C6 $\langle K_{m+2} - e \rangle = K_{m+2}$, and we have that K_{m+2} is a cycle in $\mathcal{A}_m$. The same kind of argument shows that, in fact, if $|V(E)| \geq m + 1$, then $r(E) \leq m|V(E)| - \binom{m+1}{2}$ with equality when E is complete. For a more thorough treatment, see [4].

Given a general embedding $\mathbf{p} : V \to \mathbb{R}^m$, the cocycles of an infinitesimal rigidity matroid $\mathcal{F}(\mathbf{p})$ can be described via the rigidity matrix $R(\mathbf{p})$. Using elementary column operations it can be shown that the star $S(v)$ of a vertex v minus any $m - 1$ of the edges incident with v, is a cocycle. We call these $\binom{n-1}{m-1}$ cocycles the *vertex cocycles* of v; and, for an $(m-1)$-set $A \subseteq S(v)$, we denote the vertex cocycle $(S(v) - A)$ by $S_A(v)$. In the following lemma, we see that $S_A(v)$ is a cocycle in any m-dimensional abstract rigidity matroid. Later we will use these vertex cocycles to characterize m-dimensional abstract rigidity matroids.

LEMMA 1 *Let (V, K) be the complete graph on n vertices for $n > m + 1$; let $\mathcal{M}$ be a matroid on K and consider following three conditions:*

a. For each $v \in V$ and each $(m-1)$-set $A \subseteq S(v)$, $S_A(v)$ is a cocycle of $\mathcal{M}$.

b. No cycle of $\mathcal{M}$ contains a vertex of valence less than $m + 1$.

c. $K(U)$ is independent for any $U \subseteq V$ with $|U| \leq m$, and each m-valent 0-extension of an independent set of $\mathcal{M}$ is also an independent set of $\mathcal{M}$.

Then condition a implies condition b, condition b implies condition c and if $\mathcal{M}$ has rank $m|V| - \binom{m+1}{2}$, condition c implies condition a.

PROOF:

$(a \Rightarrow b)$ Let $E \subseteq K$, let v be a vertex of valence m or less in $(V(E), E)$. Choose an $(m-1)$-set A in $S(v)$ so that all but one edge of $S(v) \cap E$ is in A. Since $|E \cap S_A(v)| = 1$ it follows that, if S_A is a cocycle, then E cannot be a cycle.

$(b \Rightarrow c)$ All vertices of all subset of $K(U)$, where $|U| \leq m$, have valence less than $(m + 1)$, so $K(U)$ can contain no cycle and is independent. An m-valent 0-extension of an independent set E contains no cycle, since, by b, that cycle would have to actually be a subset of E.

$(c \Rightarrow a)$ For this we assume that $\mathcal{M}$ has rank $m|V| - \binom{m+1}{2}$. Let v be a vertex and A be an $(m-1)$-subset of $S(v)$. The valence of v in any basis is at least m, otherwise those edges incident with v could be removed and replaced with an m-valent 0-extension to make a larger basis. Thus, $S_A(v)$ intersects every basis and so contains a cocycle. This cocycle cannot be a proper subset of $S_A(v)$ since, for any edge e in $S_A(v)$, there is a basis which intersects $S_A(v)$ in exactly that edge: let U be the m-subset of V consisting of the endpoints of the edges in A and e other than v; for each $u \in (V-U)$, including v, let F_u denote the set of m edges joining u to the points in U; finally, let $B = K(U) \cup (\bigcup_{u \in (V-U)} F_u$. Since B is formed by a sequence of m-valent 0-extensions, starting with $K(U)$, it is independent; since $|B| = m|V| - \binom{m+1}{2}$, it is a basis. Clearly, $B \cap S_A(v) = e$.

□

LEMMA 2 *Let (V,K) be the complete graph on n vertices, $n > m+1$ and let $\mathcal{M}$ be any matroid on K of rank $m|V| - \binom{m+1}{2}$ satisfying any of the conditions a–c in the previous lemma. Then, for any $U \subseteq V$ with $|U| \geq m$, we have $r(K(U)) = m|U| - \binom{m+1}{2}$. In particular, $K(U)$ is a cycle of $\mathcal{M}$ whenever $|U| = (m+2)$.*

PROOF: Let $U \subset V$ with $|U| \geq m$. Let W be any m-subset of U and let E consist of $|U| - m$ 0-extensions of $K(W)$ using the vertices in $U - W$. By condition c, E is an independent set in $K(U)$ and counting the edges of E, we see that $r(K(U)) \geq m|U| - \binom{m+1}{2}$. On the other hand, if $K(U)$ contained an independent set larger than $m|U| - \binom{m+1}{2}$, we could augment it by a sequence of m-valent 0-extensions, one for each vertex of $V-U$, to an independent set which has more than $m|V| - \binom{m+1}{2}$ edges, a contradiction. Thus, $r(K(U)) = m|U| - \binom{m+1}{2}$. Furthermore, when $|U| = m+2$, $K(U)$ is a cycle: Since, $|K(U)| = m|U| - \binom{m+1}{2} + 1$, $K(U)$ is dependent. On the other hand, for any edge $e \in K(U)$, $K(U) - e$ may be constructed by making two 0-extension to $K(U - \{x,y\})$, where x and y are the endpoints of e, and is, therefore, independent. □

THEOREM 1 *Let (V,K) be the complete graph on n vertices, $n > m+1$, and let $\mathcal{M}$ be any matroid on K. Then $\mathcal{M}$ is an m-dimensional abstract rigidity matroid on K, if and only if it has rank $m|V| - \binom{m+1}{2}$ and one of the following three conditions hold:*

a. For each $v \in V$ and each $m-1$-subset $A \subseteq S(v)$, $S_A(v)$ is a cocycle of $\mathcal{M}$.

b. No cycle of $\mathcal{M}$ contains a vertex of valence less than $m+1$.

c. *$K(U)$ is independent when U is an m-subset of V and each m-valent 0-extension of an independent set of $\mathcal{M}$ is also an independent set of $\mathcal{M}$.*

PROOF: If $\mathcal{M}$ is an m-dimensional abstract rigidity matroid on K, then, as we have seen, $\mathcal{M}$ has rank $m|V| - \binom{m+1}{2}$ and 0-extensions of independent sets are independent, hence Lemma 1 implies that conditions a–c hold.

Conversely, suppose that $\mathcal{M}$ has rank $m|V| - \binom{m+1}{2}$ and one, and hence, by Lemma 1, all of conditions a–c hold. We must show that $\mathcal{M}$ satisfies axioms C5 and C6. We first note that, since a maximal independent set of $K(U)$ can be extended by m-valent 0-extensions to an independent set containing any given edge not in in $K(U)$, that $K(U)$ is a closed set, in other words that $\langle E \rangle \subseteq K(V(E))$. We also note that the previous lemma implies that any edge set which can be obtained from a copy of K_m by a sequence of 0-extensions is a basis.

Suppose that $U, W \subseteq V$ with $|U \cap W| < m$. Since $\langle K(U) \cup K(W) \rangle \subseteq K(U \cup W)$, any edge of $\langle K(U) \cup K(W) \rangle$ not in $K(U) \cup K(W)$ must be the form (u, w) for some $u \in U$ and $w \in W$. Start with a copy of K_{m+1} which contains $U \cap W$ as well as u and w. Perform a sequence of m-valent 0-extensions to add the rest of the vertices of $U \cup W$ such that, when adding a vertex of U, the extension touches as many vertices of U already taken as possible. The result is a basis for $K(U \cup W)$ which contains the edge (u, w) as well as bases for $K(U)$ and $K(W)$, hence a basis for $K(U) \cup K(W)$. Thus (u, w) is not in $\langle K(U) \cup K(W) \rangle$ and C5 is satisfied.

If $|U \cap W| \geq m$, we can first make a basis for $K(U \cap W)$ by starting with a copy of K_m in $K \cap W$ and performing m-valent 0-extensions. Continue to perform m-valent 0-extensions to add all the vertices of $U - W$ and $W - U$ with the vertices of attachment in $U \cap W$. The result is independent and, by the edge count, is a basis for $K(U \cup W)$ which is contained in $K(U) \cup K(V)$, hence C6 is verified. □

In the above theorem, we use a global bound on the rank to specify the dimension of the matroid. In the characterization below we use a local condition to achieve the same purpose.

THEOREM 2 *A matroid $\mathcal{M}$ on the edge set of K_n is an m-dimensional abstract rigidity matroid if and only if all of the K_{m+2}'s are cycles and all of the $S_A(v)$'s are cocycles.*

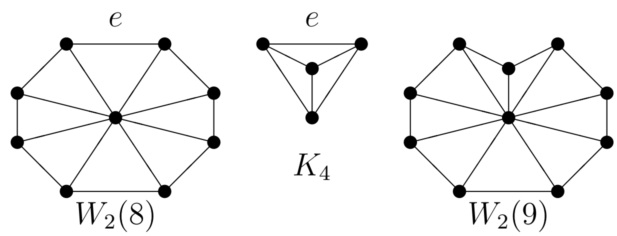

Figure 1: Wheels are dependent in dimension 2

PROOF: If $\mathcal{M}$ is a m-dimensional abstract rigidity matroid, then, by Theorem 1, all of the $S_A(v)$'s are cocycles and, by Lemma 2, every copy of K_{m+2} is a cycle.

Conversely, Suppose every copy of K_{m+2} is a cycle and all of the $S_A(v)$'s are cocycles. By Lemma 1, every 0-extension of an independent set is independent. Since every copy of K_{m+2} is a cycle, every copy of K_{m+1} is independent.

Let $l \geq 3$ and define the graph $W_m(l)$ as the join of a circuit of length l with a complete graph on $m-1$ vertices. (The join of two graphs is obtained from the disjoint union by setting each vertex in the first summand adjacent to each vertex in the second.) We call $W_m(l)$ an *m-dimensional wheel*. $W_3(4)$ and $W_3(6)$ are drawn in Figure 2. We show that an m-dimensional wheel is dependent by induction, the initial case being $W_m(3) = K_{m+2}$: We have that $W_m(l+1)$ is $(W_m(l) \cup K_{m+2}) - e$ where the K_{m+2} intersects $W_m(l)$ in a K_{m+1}, and the edge e belongs to the l-circuit, see Figure 1. Now let e be an edge of $W_m(l)$ from the l-circuit. $W_m(l) - e$ is independent since it can be constructed from the central copy of K_{m-1} by a sequence of 0-extensions. Also, adding any edge to $W_m(l) - e$ yields a graph containing $W_m(l')$ for some $l' \leq l$. Thus, $W_m(n-m) - e$ cannot be extended to a larger independent set, and so is a basis for $\mathcal{M}$, hence the dimension of $\mathcal{M}$ is $mn - \binom{m+1}{2}$. The result now follows from the previous theorem. □

COROLLARY 1 *$W_m(l)$ is a dependent in every abstract rigidity matroid of dimension m. $W_m(l)$ is a cycle in dimension m for $m = 1, 2$.*

$W_2(l)$ is a cycle in every abstract rigidity matroid of dimension 2 since every proper subset can be obtained from K_3 by a sequence of 0-extensions. By the same token, $W_m(l) - e$ is independent in any abstract rigidity matroid of dimension m if e is either a "spoke" or an edge of the "rim", however removing edges from the "hub" may leave a dependent set. For example,

$W_3(4)$, although dependent, is not a cycle in every 3-dimensional abstract rigidity matroid. $W_3(4)$ is the join of an edge e with a 4-gon. Deleting e yields the graph of an octahedron which, if $W_3(4)$ were a cycle, would be independent and rigid. Bricard [2], however, has given an example of an octahedral graph which is generally embedded and flexible.

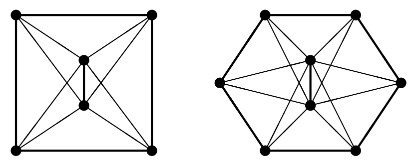

Figure 2: $W_3(4)$ and $W_3(6)$.

As stated in the abstract, these two theorems can greatly simplify the verification that a matroid is actually an m-dimensional abstract rigidity matroid. We close this note with two simple examples. First, the usual proof that infinitesimal rigidity matroids are abstract rigidity matroids involves some rather complicated geometric constructions. However, as noted above, it follows directly from the rigidity matrix that that the $S_A(v)'s$ are cocycles and it is very easy to show that the copies of K_{m+2} are all cycles. Thus, Theorem 2 gives us easily that infinitesimal rigidity matroids are abstract rigidity matroids.

Second, the simplest example of an abstract rigidity matroid which is not an infinitesimal rigidity is constructed as follows: Take the 2-dimensional generic rigidity matroid on six points and note that all copies of the complete bipartite graph $K_{3,3}$ are bases. Now add one copy of $K_{3,3}$ to the list of cycles. It follows from a result of Bolker and Roth [1], that if one of the copies of $K_{3,3}$ in an infinitesimal rigidity matriod on six points is dependent then all copies must be dependent. Thus, if we can show the result of this construction is an abstract rigidity matroid, it is the example we seek. To show that it is a matroid is straight forward. Having done that, one simply observes that the rank of the matroid has been unaltered as is the fact that condition b of Theorem 1 is satisfied. But then, by Theorem 1, this matroid is an abstract rigidity matroid.

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